Electromagnetism from Potentials

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Abstract: Maxwell's equations unify electromagnetism {"EM"} as a great milestone of theoretical physics. For the last 150 years, they have enabled practical EM calculations for science and engineering. But, in formulations of quantum mechanics and quantum field theory, it is electromagnetic potentials that are fundamental. "Force is not a primary concept (and so by extension, neither are **E** and **B**)" [Zee] . So, one ponders if even classical electromagnetism might be "conceptually easier" if one begins with potentials first. That's the usual approach for finite element analysis (FEA) of electromagnetic systems: find the magnetic vector potential first as primary and later calculate E and B if needed. The primacy of potentials is of course against the long traditions of Heaviside classical force field electrodynamics in which electromagnetic potentials are considered to be "fictitious" mathematical quantities. But the Aharonov-Bohm phase shifting effect {"AB"} forced a reconsideration of this for quantum systems.

It is suggested here that "canonical momentum," $\mathbf{P_c = mv + qA}$, and its conservation over some particle trajectories is a good starting point for a primacy of potentials perspective. For quantum mechanics, the wave operator is $P_c \rightarrow \hat{P}_c = -i\hbar\nabla$. But, kinetic momentum and "electrodynamic" momentum also determine wavelengths such as $h/\lambda = \hbar k = mv = |\mathbf{P_c - qA}|$. And the AB-effect is due to the difference in λ's for different contributing qA momenta (see Figure below). The awareness of many mathematically possible "gauge fixes" for the **A** vector potential may not be an argument against its physical reality. If the vector A-field follows current flow according to the $\vec{A} = (\mu_o/4\pi)q\vec{v}/R$ "drag equation," then the Lorenz gauge condition is merely true and is not an imposed constraint.

Introduction:

As a primary foundation for classical electromagnetism, one may begin with the basic Coulomb potential field $\phi(r) = q/4\pi\epsilon_o r$ for a point charge, q, at static rest. The electric potential field exists, it is long range and falls off with 1/r distance.

Or, from Maxwell's Gauss's Law, $\nabla\cdot D = \rho$ where $D = \epsilon_o E$ and $E = -\nabla\phi$ (scalar potential).
From the divergence theorem, $\int_v \nabla\cdot E\, dV = \int_s (E\cdot\hat{n})dS = 4\pi r^2 E = \int \rho dV/\epsilon_o = q/\epsilon_o$.
So, $E = q/4\pi\epsilon_o r^2 = -\hat{r}\partial\phi/\partial r$. And, then $\phi = +Q/4\pi\epsilon_o r$.

Let this potential ϕ be part of a 4-vector $A^\mu = (\phi/c, \vec{A}_o)$ starting with a rest case of no $\vec{A}$ field {i.e., an A_o=0 background}. Then Lorentz transform this to a moving frame: LT of A is $\mathbf{A' = \gamma(A_o - v\phi/c^2)}$.
For slow speeds, we have then derived $|A'| \simeq \mu_o qv/4\pi r$ (MKSA=SI units with $\mu_o\epsilon_o = 1/c^2$) *--electromagnetic space-time drag $\vec{A}$ is the dragging of the Coulomb field, ϕ {In texts, this formula instead derives from a wave equation for A assuming the appropriate gauge fixing}.*

When there is a distribution of charges, write $\vec{\mathbf{A}}(r,t) = (\mu_o/4\pi)\int[\vec{\mathbf{J}}(r',t')/\mathbf{R}]\mathbf{d^3r'}$, ***"Drag Equation"*** ←
where $\mathbf{R} = |r - r'|$, current density $\vec{\mathbf{J}} = \rho\vec{v}$, and charge density $\rho = dq/d^3r'$. Consider this as fundamental (that is not usually done). An $\vec{A}(r,t)$ field is dragged along with electrical current flow, $\vec{J}(r',t')$. And the existence of any background $\vec{A}$ fields is due to the presence of other background currents.

In general relativity, a similar concept is the "Lense-Thirring effect" where moving mass drags inertial frames {"gravito-magnetic field"}. In that case, the "mass current" is $J = \rho_{mass}\vec{v}$ and local inertial frames separate from that of distant stars (Lense-Thirring rotational drag is similar to a Coriolis effect from being in the "wrong" frame of reference).

QED and electrodynamics use what is called a generalized "conjugate" momentum given by $\mathbf{P_c = p_k + qA}$ where $\mathbf{p_k = mv} = m\dot{x}$ is the usual "kinetic" momentum. It is best to keep the subscript c for canonical or conjugate momentum to avoid confusions because these symbols are highly variable in literature (e.g., P for mv, p for P_c, π for p_k, …). A Lagrangian for classical E&M is $L = KE-PE = p_k^2/2m - U$ where $U = q(\phi - A\cdot v)$ – a velocity dependent potential capable of giving a "Lorentz Force" $\vec{F}=q\vec{v}\times\vec{B}$. The conjugate momentum is defined as $\mathbf{P_c \equiv \partial L/\partial\dot{x} = \partial L/\partial v = mv + qA}$. The Hamiltonian $H \equiv p\dot{x} - L = mv^2/2 + U = p^2/2m + q\phi - qA\cdot v$. For just the kinetic energy part, $H = L = p_k^2/2m = \mathbf{(P_c - qA)^2}/2m$.

Cases in which canonical momentum is conserved:

"In electromagnetism, "canonical momentum conservation" refers to the principle that the total momentum of a system, including both the momentum of charged particles and the "momentum" carried by the electromagnetic field itself (described by the Poynting vector, $p_{em}=\epsilon_0\int \mathbf{E\times B}\, d^3r$), is conserved.." [Griffiths]. That is, "the total momentum of the system remains constant over time, even when electromagnetic fields are interacting with charged particles, as long as no external forces are applied."

An "equation of motion" for electrodynamics results from applying the usual "Euler-Lagrange" equations {~ 1750} to the Lagrangian: [$\partial L/\partial x - (d/dt)\partial L/\partial\dot{x} = 0$].
But $\partial L/\partial\dot{x} \equiv P_c$, so $dP_c/dt = \dot{P}_c = -q\nabla(\phi - v\cdot A)$. That is, $\mathbf{P_c = mv+qA}$ is conserved when the gradient of $\phi - v\cdot A$ is zero {cases where potentials are uniform}. The presence of a uniform magnetic field is one of those cases that preserves P_c. The most important case considered here is the Aharonov-Bohm {"AB"} double-slit interference experiment where a long solenoid is placed just after the slits and results in an observable phase shift. There is no external B field outside the solenoid and no applied E field. Preservation of values for P_c are a key to understanding what takes place (discussed below).

So, as a second starting point for primacy of potentials, let conjugate or canonical momentum $\mathbf{P_c = p+qA}$ for an electron in an electromagnetic field and consider cases where this quantity is conserved throughout the charged particle's trajectory { an implication is that Maxwell's "qA" is indeed similar to an electromagnetic momentum -- that some form of "qA" possesses a sort of "inertia" and "reality"}. *$\vec{p} = mv$ is sometimes given the label "π" in the equation $P = \pi + qA$.*

Then, $\partial P/\partial t = \partial p/\partial t + e\,\partial A/\partial t = 0 \Longrightarrow \dot{p} = F_{em} = -e\partial A/\partial t = +eE$ {-- i.e., a force field with the usual label $\vec{\mathbf{E}} = -\partial\vec{\mathbf{A}}/\partial \mathbf{t}$}.

{For vector problems in 3d, it is better to use "total {or "material or convective"} derivatives" $F_{em} = -ed\vec{A}/dt = -e[\partial\vec{A}/\partial t + (v\cdot\nabla)\vec{A}]$ – for example the "Lorentz Force" shown below}.

The charge "e" of a particle is a coupling constant to the A field which allows the quantity "eA" to becomes an effective momentum granted to e by coupling it to the A-field. Classically, we may think of π + eA as being momenta of localized point particles.

Remember that forces take time to really act. So, even though a change ∂A/∂t occurs and causes a backwards electric field force, E; having this occur rapidly would have little effect on trajectories. Look at the case of current flow through a long straight copper wire. Before it happens, there is no neighboring B nor A field. After flow, there is a circumferential B field and an A field that is dragged along with the current in the same direction as current flow. If this happens quickly, then a particle almost stays at the same position.

$\Delta(mv) = \int q(E=-dA/dt)dt = -q\int dA = -q\Delta A$, so $\mathbf{\Delta(mv+qA)} = \Delta P_c = 0$ says that P_c = constant [StackEx].

Induction: Two Charges: $\vec{a_1}$ ← O_1 ---r ------- ⟹ O_2 → $\vec{a}_2$.

As a particular example, and picture of induction, consider two charged particles, q_1 and q_2, separated by a distance r and then accelerate the first charge by acceleration a_1: then $\partial A/\partial t = \partial([\mu_o/4\pi]q_1 v/r)/\partial t = [\mu_o/4\pi]e\vec{a_1}/r$ -- induces an $\vec{E}$ field at particle 2: $a_2 = -q_2E(r_2-r_1)/m_2$ in a direction opposing a_1. If acceleration of particle 1 is to the left, then induced acceleration on particle 2 is to the right. This point-particles case is the most elementary example of electromagnetic induction.

As usual, we may introduce the conventional label, $\mathbf{B = \nabla\times A}$ to allow for the case of rotating currents. Then $\nabla\cdot B = \nabla\cdot\nabla\times A = 0$ (i.e., div curl ≡ 0, "no-poles" equation). Create a special operator $\mathbf{D}=\nabla\times(\partial/\partial t)\_ = (\partial/\partial t)\nabla\times\_$ *(the ordering of space versus time differentiation makes no difference).*

Then $D\vec{A} = D\vec{A} = (\partial/\partial t)\nabla\times A = \nabla\times(\partial A/\partial t)$, or $\partial B/\partial t = \nabla\times(-E)$.

So, $\mathbf{\nabla\times E = -\partial B/\partial t}$ – "Faraday's Law of Induction" (with Lenz "-" sign as in $\mathcal{E} = -d\Phi/dt$ – **"Flux has an electromagnetic inertia"—**and that says something about the nature of the electromagnetic field of the Vacuum *{Faraday discovered induced current in 1831, and Faraday's law is the consequence of the equality of both orderings – space and time). Lenz law expresses a mechanism of EM inertia – an attempt at preserving magnetic fields.*

Revisit Einstein's 1905 problem of a bar magnet traveling through a wire hoop versus the picture of a wire hoop moving about the bar magnet. Consider their $\vec{A}$-fields in these two different frame perspectives. The bar magnet is equivalent to a solenoid – a cylindrical coil of current that drags an A field around it circumferentially. For the magnet approaching the wire, the mobile electrons in the wire see an increasing amplitude of A field which then induces an opposing force in the wire which tries to nullify the field changes between the magnet and wire loop. The view using "B-fields" is that increasing flux induces a back emf $\mathcal{E}$ in the loop which produces its own opposing B field. But an "A-field" view suffices by itself, and it is only relative motion that counts.

{Einstein, 1905}: "But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case."

Note that $\nabla\times E = -\partial B/\partial t = -\partial[\nabla\times A]/\partial t$ or $\nabla\times[E + \partial A/\partial t] = 0 \Longrightarrow \mathbf{\vec{E} = -\partial\vec{A}/\partial t}$ (electric field $\vec{E}$ from a vector potential, $\vec{A}$). And, of course, when there is also a scalar potential field present, we can also obtain a contribution from $E = -\nabla\phi$

A remaining "Ampere's law" of Maxwell's equations is $\mathbf{\nabla\times B = \mu_o J + \mu_o\epsilon_o\,\partial E/\partial t}$. In classical electrodynamics, this can largely be obtained from the Biot-Savart law which in turn is derived from the curl of the vector potential. In integral form, BS is $\oint_c B\cdot d\ell = \mu_o I$, or using Stokes Theorem $= \int_{sur}(\nabla\times B)\cdot dS$, where $B = \nabla\times A$ (and then some math processing).

For the field around a long straight wire, $B=\mu_o I/2\pi r$ where $I=dq/dt$ is current flow. This result (and other cases like it) may also be calculated directly using the "drag equation" but involving more calculus and algebra steps. The result is $A_z = -\mu_o I\ \ell n|r|/2\pi$ + uniform background field (*an infinitely long wire has infinite but unreal effect*). In some cases, using $\vec{E}$ and $\vec{B}$ fields in Maxwell equations simplifies calculations – but that convenience is separate from declaring these fields to be "real."

In terms of both potentials: the summary electric field is $E = -\nabla\phi - \partial A/\partial t$. Then Ampere's Law is expanded by vector identities as:

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A = \mu_o J + \mu_o\epsilon_o\, \partial(-\nabla\phi)/\partial t + \mu_o\epsilon_o\, (\partial^2 A/\partial t^2), \text{ or}$$

$$\Box \vec{A} = \nabla^2\vec{A} - (1/c^2)\, \partial^2 A/\partial t^2 = -\mu_o \vec{J} + \nabla\,[\nabla\cdot A + \partial\phi/(c^2\partial t)] \text{ --}$$

which can also be a nice wave equation **if** the term in brackets vanishes {i.e., when the Lorenz gauge $\partial_\mu A^\mu = 0$ is applied}.

Also, $\nabla\cdot\vec{E} = -\nabla\cdot\nabla\phi - \nabla\cdot(\partial A/\partial t) = \rho/\epsilon_o$, or $\nabla^2\phi + \partial(\nabla\cdot A)/\partial t = -\rho/\epsilon_o$. So, if we chose a "Coulomb" gauge $\nabla\cdot A = 0$, we would have a nice Poisson equation for electrostatics. But our focus here is the physical use of the magnetic vector potential field, $\vec{A}(x,t)$.

Gauge Choices: There are a variety of mathematical forms that overly flexible potentials can take while still yielding the same classical force fields, E and B, which can be laboratory observables. This mathematical variability might be much broader than a set of "realistic" physical potential variabilities. Mathematical calculations are usually aided by choosing appropriate forms and restrictions on potentials for a given problem; and this "electromagnetic gauge fixing" has a convoluted 150 year history. Maxwell himself preferred a "Coulomb gauge," $\nabla\cdot A = 0$. This choice is common for problems in solid-state physics, antenna theory, magnetostatics, and electrostatic problems based on Poisson's equation $\nabla^2\phi = -4\pi\rho$ {instantaneous propagation}. Relativistic electrodynamics usually prefers the "Lorenz gauge."

We could have supplemented $\vec{A}$ with a curl of another vector field ("f", physically unlikely) so that Coulomb Gauge $\nabla\cdot A = \nabla\cdot\nabla\times\vec{f} = 0$. We could have added on a gradient of a scalar "sub-potential" $\nabla\chi$ to "A" – but such "gradient fields" seem awfully unphysical (i.e., "silly") and only useful in the strange mathematical game called "Gauge Theory:"

This game includes steps of altering quantum phases at will locally, stating that this is due to modifying potential $\vec{A}$ by adding a local scalar field gradient, $\nabla\chi$, $A \rightarrow A + \nabla\chi$ -- and requiring a compensation for scalar potential $\phi \rightarrow \phi - \partial\chi/\partial t$ as well. And then, one also modifies the derivative in the Schrodinger equation from $\nabla \rightarrow D = \nabla - ieA/\hbar$ where A is a "compensating field." All of these modifications are consistently choreographed together. Although $\nabla\chi$ modifies phases, it is really the vector gauge field A that modifies quantum phase in a physically meaningful way (such as in the Aharonov-Bohm phase-shifting effect).

The Lorenz Gauge $\partial_\mu A^\mu = 0$ is used in the "Liénard–Wiechert" derivation of a common form of the vector potential starting from wave equations and yielding a more detailed retarded form of

$|A'| = \mu_o ev/4\pi R$ }. For all practical purposes (FAPP), thinking of A as the dragged electric field of a moving charge is physical and intuitive. But, to be cautious and mathematically proper, physicists avoid giving local meaning to A in the canonical summed momentum **P=p+eA** and instead integrate in closed loops to avoid gauge problems with interpretation (as in the important Aharonov-Bohm effect).

{More discussion in the Appendix at end}.

Lorenz Gauge *{1867, and often mis-spelled or mis-attributed as "Lorentz gauge"}* is:

$\partial_\mu A^\mu = 0 = \nabla\cdot\vec{A} + (1/c^2)\,\partial\phi/\partial t = 0$

Consider the simple x-axis case example with a charge, Q, at beginning at t=0,x=0 and moving as $\vec{x} = \vec{v}t$.

Examine ϕ and A at a chosen distance R from origin x=0. ϕ changes in time due to charge motion near x = 0 that shortens the distance to a point R. $\phi(R,t) = Q/[4\pi\epsilon_o(R\text{-}vt)]$, $d\phi/dt = (-Q)(-v)/[4\pi\epsilon_o(R\text{-}vt)]|_{t=0} = +Qv/4\pi\epsilon_o R^2$.

Compare this quantity against the divergence ∇·A using local variable r = R+x. $\nabla\cdot A \simeq \partial A/\partial x = (\partial/\partial x)\mu_o Qv/4\pi r = -\mu_o Qv/r^2|_{r=R} = -\mu_o Qv/R^2$ with sign opposite to that of dϕ/dt and ratio $|d\phi/dt| / |dA/dx| = 1/\mu_o\epsilon_o = c^2$.

The Lorenz Gauge is then not mysterious but is just a statement of fact if $\vec{A}$ is defined as = $\mu_o Q\vec{v}/4\pi R$ -- and this is obviously incompatible with a Coulomb gauge, $\nabla\cdot A = d\vec{A}/dx = 0$ {one has an exclusive choice of gauge}. This gauge choice allows for compatible wave equations: $\Box A = -\mu_o J$ and $\Box\,\phi = -\rho/\epsilon_o$.

For extended sources and finite propagation rate c, one can calculate the fields at distant position vector r and time t from sources at r' and t' at an earlier time = "retarded time" calculated as R = |r-r'| and t' = t -R/c.

The formula for charge conservation is $\partial\rho/\partial t + \nabla\cdot J = 0 = \partial_\mu j^\mu$ (conservation of 4-current), and the charge density at a point can change only if current of charge flows into or out of the point. Then the form $-\partial/\partial t \int \rho/R \; dvol \simeq \nabla\cdot \int \vec{J}/R \; dvol$ which can be restated as $\partial(4\pi\epsilon_o\,\phi)/\partial t \simeq \nabla\cdot[(4\pi/\mu_o)\,\vec{A}]$, or $\nabla\cdot\vec{A} \simeq -(1/c^2)\partial\phi/\partial t$.

This "Lorenz condition is fulfilled if and only if charge is conserved. It is a consequence of a property of matter and in no way an intrinsic restriction of the degrees of freedom of A^μ. Also note that "the Lorenz gauge condition is, importantly, a **Lorentz invariant** gauge condition since we're contracting the 4-indices of A^μ and ∂_μ."

The Lorentz Force [1895]: has a variety of expressions such as:

$F = I\ell \times B$, $\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$, $\mathbf{\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})} = q[-\nabla\phi - \partial A/\partial t + v\times(\nabla\times A)] = q[-\nabla(\phi - v\cdot A) - dA/dt]$,

where "total derivative" $d\vec{A}/dt = \partial A/\partial t + (v\cdot\nabla)\vec{A}$ *{i.e., over space-time, A(t,x,y,z) is a function of several variables rather than just one}.* The Lagrangian form for this case is again:

$L = ½\; m\dot{r}\cdot\dot{r} + qA\cdot\dot{r} - q\phi$ -- which has a "velocity dependent potential." Processing this via the usual "Euler-Lagrange equations" {~ 1750} yields the standard $\vec{F} = q\vec{v}\times\vec{B}$ force form {for math see https://en.wikipedia.org/wiki/Lorentz_force -- the energy $q\mathbf{v}\cdot\mathbf{A} \rightarrow \mathcal{EL} \rightarrow q\mathbf{v}\times\mathbf{B}$ force }. At first, this equation appears strange because a velocity in one direction crossed into a B field in another direction results in a force perpendicular to both. What kind of physics is that? But then note that this force is similar in form to the "Coriolis force" $\mathbf{\vec{F} = 2m\vec{v}\times\vec{\omega}}$ which is usually labeled as an "apparent force" or fictitious force in a rotating reference frame. This similarity is called "just an analogy" since the Lorentz frame is not rotating.

<u>Or is it?</u> An electron possesses both mass and charge and hence sees both the inertial world and electromagnetic space while it moves in a rotating $\vec{\mathbf{A}}$ vector background.

A velocity field (like a spinning Earth or rotating record) has $|\vec{v}_\phi| = r\boldsymbol{\omega}$. In <u>cylindrical coordinates</u>, a curl of this field is $\nabla\times\vec{v} = (\hat{z}/r)(\partial/\partial r)(rv_\phi) = \hat{z}\,(v_\phi/r + \partial v_\phi/\partial r) = \mathbf{2}\vec{\boldsymbol{\omega}}$ – the analog of $B = \nabla\times A_\phi$. So, the Coriolis force is $\vec{F} = m\vec{v}\times 2\vec{\omega} = m\vec{v}\times(\nabla\times\vec{v}_\phi$ of rotation).

A semi-uniform B field can be produced in a laboratory in the middle of Helmholtz coils (two wide co-axial coils of radius R separated height h=R). A more ideal example is the interior of a long wire-wrapped solenoid where $A_{in} = \mu_o In\rho/2 = (B_o/2)\rho \sim \rho\,\omega_B$ {ρ is radial coordinate ⊥ $\hat{z}$ and n is number of coil turns}. Both of these are cases where canonical momentum is preserved.

So, consider the simple example Case of a charged particle: e **o** -- → coordinate $\rho = \vec{v}t$ in field $\vec{A} = \otimes$.

That is, let a charged particle move radially from the origin (center of solenoid) at speed v so ρ(t) = vt. As noted before, conservation of **P = p+** q**A** means dP/dt = 0 = dp/dt +q dA/dt so that an electrical field is effectively produced, $qE = \dot{p} = F_{em} = -qdA/dt$. And, $A(t) = (B_o/2)\rho(t)$ in the circumferential or "+phi" direction.

[*as a check,* $\nabla_{cyl}\times\vec{A}_\phi = (\hat{z}/\rho)(\partial/\partial\rho)(\rho A) = \hat{z}[A/\rho\ \partial A/\rho\partial\rho] = \hat{z}\,[2(B_o/2)\rho/\rho] = \hat{z}\mathbf{B_o}$.].

So, $dp/dt = \dot{\vec{p}} = F_\phi = -qdA_\phi/dt = -q[\partial A/\partial t + (v\cdot\nabla)A] = -q(B_o/2)[v + (\partial\rho/\partial t)\cdot(\partial/\partial\rho)(\rho)] = -q(B_o/2)[v + v] =$ $-qvB_o$ in the -ϕ circumferential direction. That is, $\vec{\mathbf{F}}_\phi = +q\vec{\mathbf{v}}_\rho \times \mathbf{B_o}\hat{\mathbf{z}}$ **.** [Lorentz Force law] Since a force is produced, the kinetic momentum **p =** m**v** is not conserved.

<u>No explicit Maxwell force-field equations are required</u>.

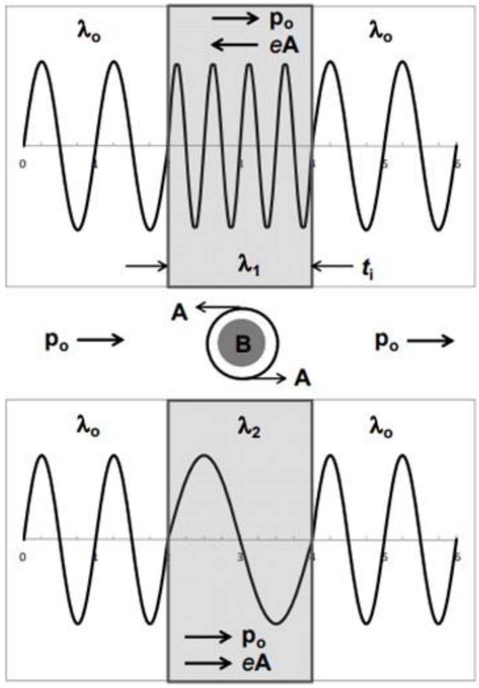

Figure 3 – The effects of the vector potential **A** on the de Broglie wavelength of an electron with an initial momentum $\mathbf{p_o}$ depend on the vector sum $\mathbf{p_o} - e\mathbf{A}$. For a magnetic field **B** pointing out of the page, this sum shortens $\lambda_1(A)$ for ψ_1 on the upper half of the solenoid, and lengthens $\lambda_2(A)$ for ψ_2 on the lower half.

[**Figure** from [Kasunic] for λ changes in A fields].

Use in electrodynamics in Quantum Mechanics:

The canonical momentum is promoted to an operator $P_c \rightarrow \hat{p}_c = -i\hbar\nabla$. But the kinetic and electromagnetic momentum also determine wavelengths like $h/\lambda = \hbar k = mv = |\mathbf{P_c - qA}|$.

A foundation for quantum waves begins with the fact that each particle mass has an associated frequency $f_o=m_oc^2/h$ and that the de Broglie relation $p = mv = h/\lambda$ results from a Lorentz transformation of this non-propagating rest vibration for a relatively moving frame of reference. These represent space-time inertia of mass.

So, how does one account for the **Aharonov-Bohm effect {"AB"}** in which the vector potential does alter wavelengths (see **Figure** [Kasunic]). An electron wave passes through two slits into two rays (#1 and #2) and then around a tiny solenoid which has a rotating A field but no external B field. The rotating A field is with the momentum of one ray but against the momentum of the other leading to a differential phase shift and a change in peak locations on a detector screen.

In free space, the electron has momentum mv_o with wavelength λ_o. In the neighborhood of the solenoid, we have a different momentum, $mv = h/\lambda(A)$. This is due to conservation of the canonical momentum so that $P_c = mv_o + qA_o \rightarrow mv_A + qA$ is preserved (and A_o far away is zero).

Hence $\mathbf{mv_A = mv_o - qA}$. Then,

$\lambda_o/\lambda_A = (mv_o-qA)/mv_o = n(A)$ as an effective index of refraction near field A.

Let $\Delta n = n_2 - n_1$, $\Delta A = A_2 - A_1 = 2|A|$, $\Delta\phi$ = phase difference between the two rays, and L = effective length across the A field of the solenoid (actually dA/dx varies smoothly rather than as a step in a rectangular region). The combination of length for the two paths is $L_1+L_2 = 2L$.

$$\Delta n = (mv_o - eA_2)/mv_o - (mv_o - eA_1)/mv_o = q\Delta A/mv_o = q\lambda_o\Delta A/h.$$

Then $\Delta\phi_{AB} = k_o\Delta x = (2\pi/\lambda_o)L\Delta n = 2\pi\, qL\Delta A/h = qLA/\hbar \sim (q/\hbar)\int A\cdot d\ell$.

The AB phase difference for a "non-integrable" phase (i.e., depending on the particular paths chosen which are not allowed to enter inside the hard central solenoid). The AB phase results from preservation of the canonical momentum through the system of slits, solenoid, and screen.

The usual AB derivation starting with the EM Schrodinger-equation gives an $e^{i\Delta\phi}$ factor without mentioning λ. But $\int qA\cdot d\ell/\hbar$ is like a "pdq" action with $2\pi\, d\ell/\lambda$ wave phase. So, an inference is $\lambda' = h/qA$ which is neither the $\lambda_o = h/mv$ nor $\lambda_A = h/(mv_o - qA)$. Rather $1/\lambda' = 1/\lambda_o - 1/\lambda_A$. However, the AB phase shift is consistent with momentum qA having its own contribution of wavelength (or additive wave-number, $k=2\pi/\lambda$). That implies that a simplest derivation of the AB phase $\Delta\phi_{AB}$ could just begin with the assumption that momentum qA has its own effective wavelength.

Simplistically, a particle with wavelength λ encounters an increasing valued $\vec{A}$ field. Intuition says that this is like a new stream current and should widen λ (but not quite the right physics). That may happen but with longer reason: P should be constant throughout a free trajectory, so increasing qA should decrease p = mv momentum (as if a backward emf were generated against p). Then λ = h/p says that λ is indeed longer because p is smaller. So the particle has not sped up in a streamline, but λ is indeed longer [as in the Figure].

Again, the standard text book approach for AB involves the time-dependent Schrodinger equation for an electron in a vector potential {in S.I. units}: $i\partial\psi/\partial t = (1/2m)(-i\hbar\nabla - qA)^2\psi + V\psi = (\hat{T} + V)\psi$ by assuming $\psi = \psi_o e^{i\phi}$ with phase shift $\boldsymbol{\phi = (q/\hbar)\int A\cdot d\ell}$ without mentioning canonical or mechanical momentum or even wavelengths themselves.

References:

[Zee] A. Zee, Quantum Field Theory in a Nutshell, Princeton, 2003.

[**Kasunic**] Keith J. Kasunic Magnetic Aharonov-Bohm effects and the quantum phase shift: A heuristic interpretation https://arxiv.org/pdf/1812.01724 **Am. J. Phys.** 87(9), 745-751 (2019) [Note: He is still referring to conservation of $P_c = mv+eA$ as $m\Delta v = -e\Delta A$ and $\lambda(A) = h/|p_o - eA|$ in papers from 2019 and 2023 – without any changes or corrections to his thesis. And the American Journal of Physics is careful in vetting its content. This is a key reference for understanding the AB phase shift effect.

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Appendix:

Gauge, In More detail:

The Lorenz gauge is expressed as $\partial_\mu A^\mu = 0$ and is Lorentz invariant. The Coulomb gauge is $\partial_i A^i = 0$ which goes along with $\phi = 0$ which is suitable for radiation problems (no nearby or distant charge sources). This is also called the transverse gauge or radiation gauge where photons are purely transverse radiation; i.e., $\nabla\cdot\vec{A} = 0$ agrees with $\vec{k}\cdot\vec{A} = 0$ in Fourier space. The Coulomb gauge separates scalar and vector potentials for individual use which simplifies calculations. It gives coupled equations $\nabla^2\phi_c = -\rho/\epsilon_o$. and $\square A_c = -\mu_o J + \nabla(\partial\phi_c/c^2\partial t)$ – where the latter term makes calculations for A more difficult using this gauge. The Poisson equation implies **instantaneous** ϕ_c but allows for properly retarded A. In the Lorenz gauge, we have $\partial_\mu\partial^\nu A^\nu = \mu_o j^\nu$ which lacks that extraneous term so that calculations are more straightforward. 'The scalar potential propagates at infinite speed, while the vector potential propagates at speed c in free space."

The Coulomb Gauge "poses challenges due to the loss of Lorentz invariance and is not optimal for problems involving relativistic particles or fields propagating at near light speed." Using ϕ_c and A_c do give the proper retarded electric and magnetic fields" since E and B don't depend on choices of potential gauges. It is a convention to specify causal retardation as brackets such as $[J]_{ret}/R$ where $J(x',t')$ uses $t' = t - R/c$ and $R = |r - r'|$ with primes for local source parameters. The Coulomb gauge "is also limiting in that although, only transverse polarizations are physical; longitudinal and scalar polarizations participate as virtual particles in interactions". "Calculating QED interactions in the Lorenz gauge is most general and should reduce to the results obtained in the Coulomb gauge after selecting a particular reference frame."

One article added, "We suggest that the Lorenz-gauge potentials may be interpreted as physical quantities."

There is a slow change in teaching methods for electrical engineering towards the primacy and ease of potentials instead of E and B fields [e.g., Carpenter]. "The underlying assumption is that ϕ and A are defined in accordance with the Lorenz gauge so that they can be treated as if propagating independently of each other in empty space." Another author agrees and adds, "The uniqueness of the solutions of the Maxwell equations is provided if it is accepted that the only gauge that can be used in electrodynamic calculations is the Lorenz gauge" [Onoochin, 2024]. Antenna theory can also be simplified [jocet].

Quantum mechanics usually avoids force equations, and Maxwell's equations essentially provide an "averaging over" the more detailed but intricate quantum phenomena. The potential, ϕ , may be physically interpreted as the potential energy per unit charge, and $\vec{A}$ as the potential energy per unit of current – and energy is the "king of concepts in physics." But, electromagnetic potentials are not physical observable – they provide another example of things that may be "real" without being directly "observable." That presents a challenge to the dependence on the scientific method that depends on experimental observations. Is it possible to reliably go one step beyond tangible observations – perhaps by a consensus deduction? Historically, it is often difficult to form a consensus on theories even using reliable experimental data. Feynman had a definition: "A real field is a mathematical function we use for avoiding the idea of action at a distance." [https://www.feynmanlectures.caltech.edu/II_15.html]. That is, "A real field is then a set of numbers we specify in such a way that what happens *at a point* depends only on the numbers *at that point*." The Aharonov-Bohm effect was a big surprise to some people in demonstrating an observable shift in interference peaks from the presence of an $\vec{A}$ field.

Can we dispense with B and E force fields and stay with just potentials? Maybe, but our history and literature have highly valued these derivative fields. I've always puzzled about E an B fields being real. We say they are because energy is real, and E^2 or B^2 represents energy. But basic reality may be just A-field dragging from moving charges. So, are hidden mechanisms in spacetime performing $\nabla\times\vec{A}$ and -∂A/∂t operations by itself all the time – or only when interacting with other charges? It's not clear. Voltage kicks from inductor discharge of stored energy suggest B field reality by itself (no interactions with external charges required). Maybe spacetime itself notices A field stresses from space or time "curvature of A(x,t)" and recognizes that as energy density. The "Vacuum" itself does contain an understanding of EM fields (that we label by A_μ). So mathematical processing by spacetime isn't inconceivable.

Appendix Note on *Four Vectors:* $P = m_o U = (E/c, p)$, $A = (\phi/c, a)$, $K = (\omega/c, \vec{k})$, $X = (ct, \vec{x})$, $\boldsymbol{\partial} = (\partial_t/c, \nabla)$.

A scalar or inner product of two four-vectors $A\cdot B = A_\mu B^\mu$ is invariant under *any* smooth coordinate transformation. A relevant example is phase:

$K\cdot X = \pm(\omega t - kx)$ for metric signature (+ - - -) or signature (- +++). [or $g_{oo} = \eta_{oo} = +1$ or -1].

The complex exponential plane wave is $\psi(x,t) = \exp[i(kx-\omega t)] = \exp[i\, K\cdot X]$

For the direction of wave motion, signature doesn't matter: that is, if we look at any particular wave peak (fix the phase), then $\pm(kx-\omega t) = 0 \Longrightarrow x = \omega t/k = v_{phase}\, t$ (in the positive right direction).

The de Broglie relation $p = h/\lambda = \hbar k$, and $E = \hbar\omega$ has $\mathbf{P = \hbar K}$ as 4-vectors.

with $\psi = Ae^{+i\phi}$, the standard form $\varphi = (kx-\omega t) = ([px-Et]/\hbar)$. If we have an operator $\hat{p} = -i\hbar\partial/\partial x$, then $\hat{p}\psi = -i\hbar\,(ip/\hbar)\psi = p\psi$, eigenvalue equation {and $\partial/\partial x = +i\hat{p}/\hbar$ }. But with the other $-i\phi$ form, we would need $\hat{p} = +i\hbar\partial_x$. Standard QM uses the operator $\hat{\mathbf{p}} = -i\hbar\partial/\partial x$ or $-i\hbar\nabla = -i\hbar\nabla_i$ with phase = kx-ωt (and we could say $g_{oo}=-1$ convention).

SI Electromagnetic UNITS [ ]

$[\mu_o] = 4\pi \times 10^{-7}$ H/m = 1.257×10^{-6} N/A^2 = vacuum permeability

$[\epsilon_o]$ = permittivity = $1/\mu_o c^2$ Farads/m = $C^2s^2/kg\ m^3 = 8.854 \times 10^{-12}$, so

$1/4\pi\epsilon_o = 9 \times 10^9\ Nm^2/C^2$ (or 8.9877)$\times 10^9$ kg m^3/C^2s^2.

Potential ϕ = V (volts) = $\mathbf{(1/4\pi\epsilon)Q/r}$ kg m^2/Cs^2, $E=-\nabla\phi$, [volt] = W/A = kg $m^2/A\ s^3$ = joule/As .

Vector Potential $[\vec{A}]$ = [p]/Coulomb, so [eA] = [p] = [kg m/s], $\vec{\mathbf{A}} = \mu_o\,\mathbf{Qv/4\pi r}$

[A] = volt sec/meter = joules sec/Coulomb· m. $|\vec{A}|/\phi = v/c^2$ for the same r and Q.

In electromagnetism, current density is the amount of charge per unit time that flows through a unit area of a chosen cross section. [J] amps/m^2 = C/sm^2 $dq = \rho v dt dA$, $j = \rho v m$, $\nabla\cdot J = -\partial\rho/\partial t$.

$P = \pi + qA$: More on foundational conjugate momentum where usual $\hat{p} = -i\hbar\partial/\partial x \rightarrow \hat{P}_c = -i\hbar\partial/\partial x$.

In electrodynamics, we introduce the conjugate or generalized momentum $P = \partial L/\partial v$ using the Lagrangian $L = \frac{1}{2}mv^2 - q(\phi - v\cdot A)$ with velocity dependent potential; so $\mathbf{P \equiv p_c = mv+qA = \pi + qA}$. Then, kinetic mv momentum is $\boldsymbol{\pi} = \mathbf{p_c - qA}$. The Hamiltonian in an external potential is then just $H = \pi\cdot\pi/2m + q\phi$. As operator, $\hat{\pi} = -i\hbar\nabla + qA$, or

$\nabla_{cov} = (i/\hbar)(\hat{\pi} - qA)$ which is also called a "covariant derivative" D_i {as the spatial part of $D_\mu = (D_t/c, \nabla_{cov}) = (\partial_o/c + iq\phi,\ \partial_\mu - ieA_\mu/\hbar)$ }. D includes kinematic and electromagnetic momentum.

The Hamiltonian is $H = (1/2m)(P-eA)^2 = (1/2m)\pi^2$ where $\hat{P} = \hat{p}_c = -i\hbar\nabla$, so $\hat{\pi} = -i\hbar(\nabla - ieA/\hbar)$. Note two things: energy still goes with ordinary mv momentum, and it is P that becomes a derivative operator, $\hat{P}$. That is, qA is included in general wave properties (emphasized by Mike Jones, Vol. 1 p 108).

Potential dragging from currents is similar to "Gravitational Frame Dragging."

The magnetic potential A field is a result of a Lorentz transformation of a Coulomb field. If we did the same thing to the Newtonian Gravitation potential $\Phi = MG/r$, we would get the result $A_g = \gamma v\Phi/c^2 = \gamma vMG/c^2R$ as a weak field falling off as 1/R. But, this isn't quite right. Gravitation is not a vector field but a tensor distortion of the fabric of space-time. Although both space and time are warped by nearby masses, Newtonian gravitation is due solely to just the distortion of time, $dt/d\tau$. For weak fields from moving masses and non-relativistic speeds, one needs the weak metric $h_{\mu\nu}$ with non-zero off-diagonal elements h_{io} which yields a frame-dragging result $A_g = \mathbf{-4}GMv/c^2R$. Sciama's 1953 discussion of the Origin of Inertia ignores this new factor of 4 -- which is OK since he is just demonstrating a plausible analogy. He also uses an old cosmological approximation $G\rho\tau^2 \sim 1$ where tau $\sim R_{universe}/c$. Sciama's paper was highly stimulating – but not quite right.

Einstein mentioned the inertial dragging "Lense-Thirring" effect as implying an increase in effective inertial mass due to neighboring "ponderable masses," M, using an "Einstein-Sciama" inertial force equation, $\vec{F} = 4GMm\vec{\mathbf{a}}/c^2r$. This is also "not quite right," since Brans showed that this cannot be a real effect because all local gravitational fields can be transformed away with appropriate accelerated frames of observation (the "Principle of Equivalence"). Electromagnetic induction uses a dragging of "electromagnetic space" vector potential $\vec{A} \sim \mu_o Q\vec{v}/4\pi R$ whose rate of change produces a counter electric field, $E = -\partial A/\partial t$, that can act on nearby charges to accelerate them in an opposite direction resulting in an attempt to preserve the summed values of the A field (Lenz's Law). Einstein induction would be an opposite of this. Accelerating a local mass should induce a local force field that might accelerate other local masses in the same direction. Presumably, this doesn't happen in the linear case – but it might apply to rotating Kerr black holes. And, when a mass current exists, space oriented with respect to distant stars feels as if it were undergoing a rotation in the direct of dragged A_g. This can effectively result in Coriolis type forces from "being in the wrong frame of reference" {"gravito-magnetism"}.

Modern physics grants a higher level of "reality" to concepts that are Lorentz invariant. Momentum and kinetic energy are relative, but $E^2 - (pc)^2 = (mc^2)^2$ is invariant. However, observed phenomenon are usually relative and still real to us. Particle wavelengths don't exist unless particles are moving relative to us. Vector potentials require relative motion or current flows, and induction requires changes in current flows. Should we dismiss phenomena because they can be transformed away using special moving observers? The term "real" is not well defined.