Nobel Prize for Topology in Exotic Materials.

David Peterson, 10/9/16 - 11/24/16 - 12/31/16 Revision 1.

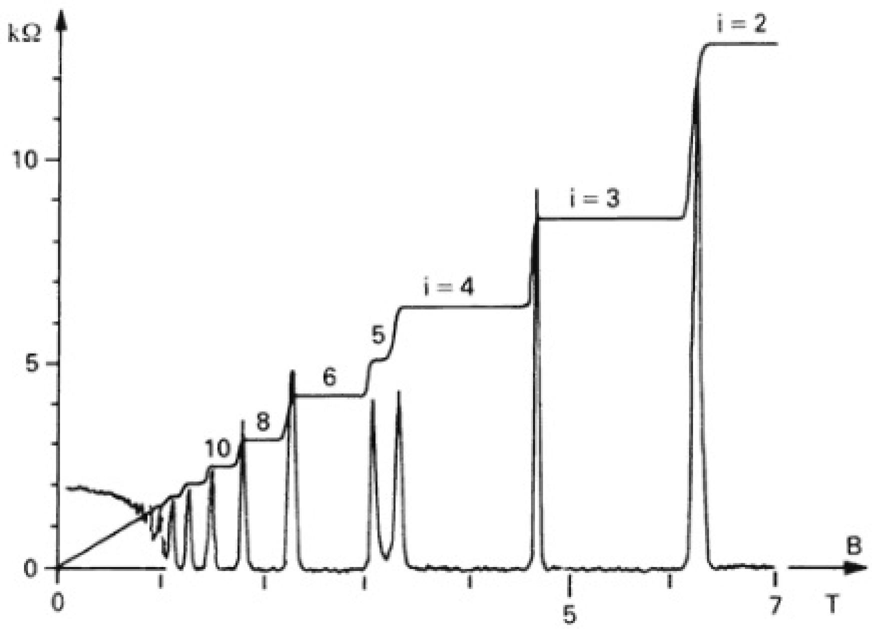
On October 4, 2016, the Nobel Prize in physics went to Thouless, Kosterlitz, and Haldane “for theoretical discoveries of topological phase transitions and topological phases of matter.” Half of the prize was given to David Thouless for two key advances: In the early 1970’s it was believed that superfluidity and superconductivity were not allowed for very thin 2D layers. Kosterlitz and Thouless showed that wasn’t true with the use of topological concepts. And then in the 1980’s, Thouless helped explained the mysterious “Integer Quantum Hall Effect” again using topology and “marked the discovery of topological quantum matter.” Since then, condensed matter physics of topological materials has blossomed!
[Nobel prizes were awarded on December 10, 2016. But Thouless has not yet presented his work yet. Hopefully he will submit an essay sometime in 2107.]

This note mainly focuses on the **Integer Quantum Hall Effect** (“IQHE” or just QHE) [e.g., see **Figure 1** below]. One author declared in general, “The quantum Hall effect (QHE) is one of the most remarkable condensed-matter phenomena discovered in the second half of the 20th century. It rivals superconductivity in its fundamental significance as a manifestation of quantum mechanics on macroscopic scales.” [5] It “is now used to maintain the standard of electrical resistance by metrology laboratories around the world” and measures the fine structure constant alpha accurately to 10^{-8}.

Typically, IQHE needs a two-dimensional electron gas (2DEG), and that can be formed in a thin layer of semiconductor next to an insulator (called an “inversion layer” , e.g., AlGaAs on GaAS). The thickness of this gas may only be 30 angstroms but still can form a broad holistic layer over the relatively large semi-rectangular Hall probe area. The quantum Hall effect is macroscopic! Temperatures < 1 kelvin and magnetic fields > 10 tesla are often also needed [but graphene can show effects at room temperature]. Applied voltages in the long x direction of a rectangle cause a build-up of voltage in the y width direction, so conductivity technically needs be a 2D tensor: $J_i = \sigma_{ij} E_j \ (includes\ J_x = \sigma_{xy} E_y)$. Hall resistance is measured in the cross y direction and was observed to change in integer steps on plateaus .
$\sigma_{xy} = \nu e^2/h \ or\ \varrho_{xy} = -h/\nu e^2 = R_K/\nu$. $R_K = h/e^2 \sim 25.6\ k\Omega$ is called the “von Klitzing constant (and is good to 9 figures). [note that the fine structure constant is
$\alpha = e^2/4\pi\varepsilon_o \hbar c\ [SI],\ so\ R_K\ determines\ \alpha$]. A requirement for topological integer plateaus is having imperfect materials (doping ions, surface roughness, random disorder -- and most materials do have uncontrolled imperfections). Large magnetic fields are needed to see the biggest “ground” plateau. And, note that (with the right setup) going to 30 T may introduce an unexpected “fractional plateau” (1/3rd) -- a separate and very weird arena with largely different physics (see Fractional Quantum Hall Effect FQE in a section below).

Details of the IQHE are intricate, dovetail in an almost conspiratorial way, and are difficult to the point of first requiring reading an entire book on the subject (such as that of David Tong, [1]). Robert Laughlin (Nobel 1998) would insist that this new physics is “emergent” from collective phenomena and can not be mathematically deduced from fundamental physics (the whole is greater than its parts). Others will try anyway but with some mystery and opaqueness.

Before a more detailed view of all this can be discussed, it is necessary to first introduce several preliminary topics: the standard Hall effect in classical physics, Topology, Landau levels, Anderson Localization, Fermi levels, and Edge modes.



[Source: K. v. Klitzing, G. Dorda, M. Pepper: *New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance*, Phys. Rev. Lett. **45** (1980) 494-497, Figure 1.]

Figure 1. IQHE Plateaus shown by quantized electrical resistance Rxy versus applied magnetic field B and labeled by Landau level integers, i. Continuing B to above 10 T would reveal the i=1 plateau. The Landau Level (LL) spikes are for direct lengthwise resistance Rxx > 0 and occur at the transitions between plateaus. On the plateaus, Rxx = 0.

Classical Hall Effect:

Every freshman physics text presents the classical Hall effect of 1879. If a current, I_x , flows through a thin metallic strip that has a strong perpendicular magnetic field, B, going through it, then a potential difference develops between the sides. Some density of charge carriers, n, in the strip flows with a slow drift speed v and experiences a cross field force

$F_y = qv_x \times B_z$ *with current* $I_x = ne\delta wv_x$ where delta is thickness (very thin) and w is cross width in the y-direction. Then,

$$F_y = ev_xB = BI_x/new\delta. \text{ But } V = Fw, \text{ so, } V_y = I_xB_z/ne\delta .$$

$$\text{And resistance } R_{xy} = V_y/I_x = B/ne\delta \qquad \textbf{Eqn. 1}$$

The formula says that even to get micro-volts of voltage will require high magnetic fields (like $B \geq 1\ T$ where tesla = 10,000 gauss in current college labs) and extreme thinness ($\delta \sim 20$ microns or less).

The American physicist Edwin Hall used thin gold leaf for his conducting strip and revealed the effect well <u>before</u> the discovery of the electron. So what he revealed was that the quantity “ne” flowed through the gold as a negative current. Positive current flow would give an opposite side-to-side voltage (good for semiconductor positive hole flow). Using E as the induced electric field sideways and J is the current flow density through the strip, a “Hall <u>coefficient</u>” was defined as: $"R_H" = E_y/J_xB_z = V_y\delta/I_xB = -1/ne$ **[Eqn. 2]** showing a way to measure carrier density or magnetic field B (“Hall effect probe). Note that this unfortunate naming convention is different from the Hall <u>resistance</u> above, so resistance is $R_{xy} = B\ R_H/\delta$. The rewards of this classical measurement are knowledge of charge density for carriers and resistivities for materials. And in the 20th century, carriers could be “holes” with an effective positive charge. One should also study the “Drude Model” which adds a friction term to cyclotron motion in the form of a scattering time, τ . It is this model that makes clear that conductivity should be treated as a 2x2 tensor leading to resistivities:
$\varrho_{xx} = m_e/ne^2\tau$ *versus the usual* $\varrho_{xy} = B/ne$. And when we find that
$\varrho_{xy} \neq 0$, *then* $\varrho_{xx} = 0 \Longrightarrow \sigma_{xx} = 0$! (unexpectedly the system is then a perfect insulator). The longitudinal Rxx depends on sample composition and sample length.

<u>Topology:</u>

We tend to think of topology as the counting of “holes” through geometric objects (something through which a string can thread); and for one hole, we consider a coffee cup to be “the same as” a donut. The number of holes represent “topological invariants” that are usually integers. But, the term “hole” can also apply to objects of any dimension. So, for example, the inside of a sphere is called a 2-hole (something that can be filled with water). There are also a variety of types of topological indexes and other concepts that are hard to picture.

One goal of topology is to identify properties of objects that are invariant under continuous deformations. A simplest example of a topological concept is that of “deformation classes” or “path components” of geometric regions, S. This means that for any two points, there can be a continuous path ending on the points, and this idea obviously applies to a 2-sphere, or a torus surface, or infinite Euclidean spaces E^n . The symbol $\pi_o(S) = 0$ is used for the set of all path segments that can be deformed into each other. A virtue is that “global topological properties are robust against local perturbations [7].”

But, if there is a forbidden “**gap**” separating two materials, then there is no continuous path between them. The idea of a forbidden barrier also applies to physical “phases” so that solid ice is separate from water fluid (liquid/gas) on a pressure versus temperature plot (a path does exist between liquid and steam by going around the “triple point”). There is a “phase transition” between between solid and fluid states. We now know that there are other types of phases in condensed matter physics such as topological superconductors, topological insulators, superfluidity, and now the quantum Hall states. In IQHE, there is a phase transition at specific energy levels so that a normally insulating material suddenly becomes a good conductor.

The role of topology in condensed matter physics often enters through quasi-momentum on the **“Brillouin torus.”** For crystals, electron states depend on the geometry of the lattice which generally repeats from atom to atom. The potential energy is periodic like the lattice, and the wavefunction is also periodic: $u(x + na, y + nb) = u(x, y)$ for a rectangular lattice. The primary difficulty is dealing with the vast variety of possible types of crystal structures. Including momentum gives a “Bloch wave:” $\psi(r) = e^{ik \cdot r} u(r)$ where k is the crystal wave vector and momentum $p = \hbar k$.

The simplest rectangular physical lattice has another view called the “reciprocal” lattice with primitive cell sides: $A = 2\pi/a \text{ and } B = 2\pi/b$ which is effectively a Fourier transform of a simple physical lattice. Reciprocal lattice points or vectors G in this Fourier space are $G = hA + jB$ where h and j are integers. Crystal wave diffractions are satisfied when $\Delta k = G$. A cell of size A x B is called a “first Brillouin zone.” Because of periodicity, the opposite sides are “identified”, and that means homeomorphic to a <u>torus</u> ($T^2 \text{ for } 2d \text{ and } T^3 \text{ for } 3d.$). “The fundamental group” for the torus is: $\pi_1(T^2) = \pi_1(S^1) \times \pi_1(S^1) = Z \times Z$, where Z is the set of integers (e.g., representing “winding numbers” about a circle).
This means that there could be non-contractible <u>loops</u> (rather than the previously mentioned arcs) around the torus representing many integers of winding numbers. [Note that S being “simply connected” implies that $\pi_o(S) = 0 \text{ and } \pi_1(S) = 0$]. “The full ensemble of states over the Brillouin torus is always trivial.” But an energy gap can cause a split into two well separated sub-ensembles each with non-trivial topology. This is related to Thouless’ original Chern topological index. “The Chern number is topological in the sense that it is invariant under small deformations of the Hamiltonian.” [21]

As a short hint on these topics: Chern number, Berry phase, and classical “Gauss-Bonnet” Euler characteristic can all be calculated as integrals.

$$\chi(sphere) = (1/2\pi)\int_M K dM = \chi = 2 - 2g.\ E.g.,\ \chi(S^2) = 2,\ g = 1 - \chi/2 = 0,\ and\ \chi(T^2) = 0$$ [23].

K is the “curvature” of a Manifold, g = genus = holes/handles for a 3D surface. A sphere has no handles and a torus has one hole. A simpler example is for a 2D circle

Circle S^1 : $(1/2\pi)\int_{circle} Kds = (1/r)(1/2\pi)(Cir = 2\pi r) = 1 = \chi \quad in\ 2D$.

The 3D genus and Euler characteristic also pertains to the old high-school geometry: vertices - edges + faces= V-E+F. For a 4-faced tetrahedron, 4-6+4 = 2 so g = 0 (no holes).

Berry Phase uses Stokes' theorem to get a form: $\gamma = \int_S dS \cdot \Omega(R)$ where Omega is a Berry curvature from a Berry connection and R is a vector parameter of time.

The topologically invariant Chern number, Ch_n *or* c, comes from the integration of "Berry curvature." A nonzero Chern number says that there is an obstruction in applying Stokes theorem over the entire parameter space [-- see "Geometry in Modern Physics" [6]]. If one wants to see plentiful applications of topology, condensed matter physics is the place to be -- however, the dovetailing of the Chern numbers to IQHE is acknowledged to quite difficult [9].

Many articles on topology and physics deal with the "real world." But, the topology in the quantum Hall effect is really a topology in a quantum state" and quantum topology is now used for many application. "Berry phase is the simplest demonstration of how geometry and topology can energy from quantum mechanics" and at the heart of the IQHE. This phase shift occurs when a complete loop is made in some parameter space and is geometric and separate from the usual Edt and kdx phase contributions. The leading example is the: <u>Aharonov-Bohm (AB)</u> effect with phase change $\gamma = \oint_C eA_i dx^i$ (e.g., for a closed path around a solenoid). And this is applied below.

In modern condensed matter experiments, one can additionally see analogue cases of formation of Dirac monopoles and also Yang monopoles with non-vanishing 2nd Chern number measured for the first time [7]. A research article by NIST said: "Fundamentally, topological order is generated by singularities called topological defects in extended spaces, and is quantified in terms of Chern numbers, each of which measures different sorts of fields traversing surfaces enclosing these topological singularities. Here, inspired by high energy theories, we describe our synthesis and characterization of a singularity present in non-Abelian gauge theories - a Yang monopole - using atomic Bose-Einstein condensates …"

Topological materials have topological properties that are "robust and insensitive to perturbations and impurities." They "stay the same if you continuously change the system: stretching it, straining it, shaving off some layers – or really any change that doesn't cause a phase transition." [17]

Claimed definitive explanations of IQHE can be shown in several different ways; and one seems to require "Non-Commutative Geometry," [Bellissard, 1994, ref. [3]]. Hall conductance is a non-commutative Chern number, "Ch." That is, $\sigma_{xy} = \nu e^2/\hbar = (e^2/h)\, Ch(P_F)$ interpreted as a Chern character from a "Kubo-Chern" relation. The first inroad to understanding IQHE quantization was given in a famous (*-ly undreadable*) 1982 paper referred to as "TKNN" [8] for its four authors (one of them being David Thouless). It says "Hall conductance is quantized whenever the Fermi energy lies in an energy gap, even if the gap lies within a Landau level."

Landau Levels:

A first step is to talk about electron motion in a thin film with a normal magnetic field, B. The Lorentz force F = qvB will be balanced out by "centrifugal" force $F = mv^2/r$ where $v/r = \omega = angular\ motion$. The electron will go around in circles with a "cyclotron" frequency" $\omega_c = qB/m$ where m is the effective mass of the electron. Since Boltzmann's constant is $k_B = 8.61 \times 10^{-5}\ eV/K$ and lab temperatures are below 1 K, thermal fluctuations have negligible effect. This allows for the emergence of quantum effects such as quantized Landau levels and quantized magnetic flux. A typical energy for $\hbar\omega_c \sim 10\ meV\ for\ fields\ B \sim 10\ T$. An analogy with old Bohr, one aspect of circular motion is that a circumference has to be integer multiples of wavelengths round the circle.

Mathematical Derivations:

The presence of a magnetic field in a z-direction alters a term in the Hamiltonian as $H = (p - qA)^2/2m$ as if a vector potential A times charge acted as "electromagnetic momentum" The term (p-eA) is called "canonical momentum," as opposed to usual "mechanical momentum" $p^{\mu}{}_{mech} = m\,\dot{x}^2$. The vector potential is not gauge invariant, and Lev **Landau** picked a special "Landau gauge" for A: $A_x = -By$ (or alternately $A_y = Bx$ with all other $A_i = 0$) which acts as a simple shearing field indeed giving $\nabla \times A = B$ as it should. Then the Hamiltonian could be written as
$H = [(p_x + eBy)^2 + p_y{}^2 + p_z{}^2]/2m$. If we were to label an "offset" distance as $y_o = -p_x/eB$, we could write out a term, $[m\omega_c{}^2(y - y_o)^2/2]$, exactly matching the first term above (one has to expand both squares and match up the terms). The this second degree of freedom is the coordinate of the center of the cyclotron orbit.

Now, the standard "Linear Harmonic Oscillator" **(LHO)** has a similar form
$H = p^2/2m + m\omega_c{}^2y^2/2$ where the last term incorporates a vibrating spring energy. For IQHE, we have a $(y - y_o)^2$ term instead of a y^2 term implying a new off-centering concept. This displacement can be thought of as $y_o = kl^2$ where l is "magnetic length"
$l = \sqrt{\hbar c/eB} = 25.7nm/\sqrt{B\ teslas}$. In the IQHE, B includes many magnetic flux quanta
$\Phi_o = h/e \sim 4 \times 10^{-15}\ Wb$ -- webers a unit of magnetic flux (or half that value for the case of Cooper pairs for superconductivity vortices) so that the density of magnetic flux is
$B = \Phi_o/2\pi\, l^2$.

Using these Hamiltonians in a quantum mechanics setting requires solving the Schrodinger equation where H is treated as an operator: $\hat{H}\Psi = E\Psi$. We don't have to do that

here because all standard texts solve the easier LHO problem and present its wavefunctions. We then know that the quantum linear harmonic oscillator ends up having quantized energy levels according to the famous formula: $E = (\nu + 1/2)\hbar\omega$; and because the Hamiltonians are similar, that will also apply to the energies for circular motion Landau Levels. So energy could be pictured as increasing in steps of
$\nu = 0,\ E_o = \hbar\omega_c/2,\ \ and\ then\ \nu = 1,\ E_1 = 3\hbar\omega_c/2,\ and\ \ \nu = 2...\ So\ \Delta E = \hbar\omega_c \sim 10\ meV$ is the gap separation energy.

So, electrons may ideally only occupy orbits with discrete energy values.
And, the n above determines the integer n in the IQHE! The Landau level location are where the IQHE makes its ϱ_{xy} jumps in cross resistivity, and the "spikes" in Figure 1 represent direct resistivity ϱ_{xx}. These are also peaks where the Landau "density of states" [DOS = g(E)] or "degeneracy" is high. The strangest result is the occurrence of a "phase transition" of extended states at every Landau Lever band center (i.e., the "spikes").

Note that the energy here didn't depend on the $p_x = \hbar k_x$, so degeneracies can exist. If LHO eigenstates are labeled by $|\varphi_n>$, then the state of an electron can be:
$\Psi(x,y) = exp(ik_x x)\ \varphi_n(y - y_o)$ which depends on the quantum numbers n and k_x. As the n values and energy levels rise, it turns out that the now fuzzy wavefunctions increase in radial size as well [as $< r^2 > = 2(n+1)\hbar/eB$ (wider circles). And they also have angular momentum: $L_z\Psi_n = \hbar n\ \Psi_n$. This radial increase turns out to be important to the understanding of IQHE.

As mentioned before, these Landau levels can only be observed for very low temperatures and very strong magnetic fields: $\hbar\omega_c >> kT$. It is important to estimate how many sublevels can exist in a Landau level (the degeneracy of the ground state). The answer is $N \sim BL_xL_y/\Phi o,$ where L is the width of the Hall strip [5] and Phi is a tiny quantum of magnetic flux. If due to Zeeman energy splitting, it is"typically about 70 times smaller than the cyclotron energy" [9] for GaAs. The degeneracy increases with the applied magnetic field through a characteristic area. "There is one electron-state per Landau level per flux quantum." So, in tests where the B field ramps up, more electrons can go into the lower LL's. That is why the high B fields of Figure 1 reveal the low labels of the LL's.
Levels are characterized by integer called "filling factors," $\nu_f = hn/eB$ where n is the surface electron density and ν_f is "the ratio between the total number of electrons and the number of states on one Landau level." [17]

"Anderson Localization:"

In general, "electronic conductivity should be directly proportional to the electron mean free path [4] which is typically ~ 100 nm. But, in 1958, Philip Anderson wrote a complicated paper suggesting that electron scattering can be much more localized in the presence of many crystal defects. Doped semiconductors is one example of a disordered crystal lattice (acting somewhat like random potentials at crystal sites). In Anderson's electron localization, the electron zigzags between impurities resulting in a smaller mean free path and hence greater resistance. If a "localization length" is labeled as ξ, *then* $|\psi(r)|^2 \sim e^{-|r|/\xi}$. A short localization length restricts electron propagation. If motion is free across the entire Hall strip, then probability is unlocalized or "extended" and constant. In the presence of large B fields, localization is <u>different</u>; and there is only one critical energy allowing for an extended state (pretty much in the center of a DOS peak at Landau energy). Disorder broadens the DOS peaks, and anything to the sides of dead-center still is localized with only the <u>middle</u> being delocalized (a strange emergent result that is hard to understand in any simple way).

Impurity scattering dominates at very low temperatures. It happens that localization lengths <u>diverge</u> exactly at Landau levels whereas in-between these levels, direct conductivity vanishes and Hall body electrons are localized. That means that the plateaux in Figure 1 owe their existence to localization from crystal disorder. Modeling of the effects of impurities can be accomplished by using a random potential V(x) in the electron Hamiltonian [1]. Quantized resistivity persists on these precise plateaux over a range of increasing magnetic field strength, B, and charge carrier density, n.

The details of LL conductivity are very tricky and subtle. Between two adjacent Landau energy levels, there is strong Anderson localization; and localization blocks conductivity. Bulk states are insulating. Exactly at the Landau level, the localization length diverges into conductive "extended states." As one increases electron density at a Landau level, the filling gets added into the bulk localized states caused by disorder so that they don't add on to net transport (Hall conductivity is a quantized constant > 0). The conductivity $\sigma_{xy} = \nu e^2/h$ gets "stuck." In-between Landau levels, increasing the Fermi level only occupies localized bulk states. Only the narrow centers of the Landau Levels (LL's) have current carrying extended states. $\sigma_{xx} \to 0$ *and* $\sigma_{xy} > 0$, *then* $\varrho_{xx} = \sigma_{xx}/(\sigma_{xy}{}^2 + \sigma_{xx}{}^2) = 0$, zero direct resistivity too.

<u>Summarizing the above:</u>

Magnify a little part of Figure **1** to consider just one of the plateaus between a direct "spike" on the left and another spike on the right. The spike itself results from a sudden increase of "localization length" or "extended state" phase change from insulator to metal allowing a boost in conductivity so that $\xi >> 0$, $\sigma_{xx} \sim h/e^2 > 0$ *along with* ϱ_{xx} *and* $R_{xx} > 0$. In the plateau we have the emergence of fixed (stuck, persistent, quantized) non-zero resistivity and conductivity for topological invariants
ϱ_{xy} *and* σ_{xy} *but also* σ_{xx}, ϱ_{xx} *and* $R_{xx} \sim 0$. *And* $\xi \sim 0$ means strong Anderson localization.

There are now many approaches to the physics of localization including some that treat it as a <u>critical phenomenon</u> using a size varying "scaling function $\beta(g)$" -- as in quantum field theory (QFT). In 1984, Libby, Levine and Pruisken attacked the phase change problem incorporating a "theta angle" into the Anderson model [8]. This is an idea of an "instanton vacuum" and "nonlinear sigma model" borrowed from quantum chromodynamics (QCD) for quark confinement versus deconfinement. Then there is a "renormalization" flow diverging at the Landau energies and producing quantization. This means The robustness of IQHE plateaus is seen as a large scale emergence. It is rather amazing that ideas from high energy physics may pertain to solid state physics, but they are gathering experimental validation [11]. But also recall that some of these particle physics concepts originally came from Anderson's studies in solid state physics (e.g., the Higgs Symmetry Breaking idea). Unfortunately, Pruisken's field theory is qualitative and has not been able to calculate quantitative results. Numerical approaches then seem best, and the fluctuations seem to be multi-fractal in nature.

The insulator to metal transition looks like a critical point phenomenon of the form: $\xi/\xi_o = |E_o/(E - E_c)|^{2.33}$ *where* $\xi_o \sim$ *magnetic length*, E_c = *critical pt. LL*, E_o = *characteristic* energy . The power drop-off $\nu \simeq 2.33$ is a universal constant. Despite this blow-up to infinite delocalization, longitudinal conductivity is still finite e.g., $\sigma_{xx} \sim 0.54\, e^2/h$. The IQHE phase change is one of the best known examples of a quantum critical point of a disordered system,. In this case, it is a continuous phase transition or second order phase transition with zero latent heat [12].

Fermi Level:

Electrons are half integer spin fermions obeying the Pauli exclusion principle. That means that two electrons with the same quantum numbers cannot get too close to each other. The number of states per unit volume with a given energy ε_i (*electron volts eV*) and degeneracy $g_i(\varepsilon_i)$ is given by $N_i = F(\varepsilon_i)g(\varepsilon_i) = g(\varepsilon_i)/[1 + exp[(\varepsilon_i - \mu)/kT]$, *where* $F(\varepsilon_i)$ is called the Fermi-Dirac distribution, and mu is "chemical potential." The term "Fermi energy" usually refers to "the (kinetic) energy difference between the highest and lowest occupied single-particle states in a quantum system of non-interacting fermions defined as *always at an* absolute zero temperature." In a metal, the term "lowest occupied state" usually means the bottom of the conduction band.

The "Fermi level" or "electrochemical potential" in a metal at absolute zero is the energy of the highest occupied single particle state including both kinetic and potential energy (the energy of the lowest state). It is the surface of the sea of electrons such that no single electron can rise above it. So, the Fermi level is the total chemical potential for work required to add one electron to the body.

In solid state theory, atoms are packed close together so that their previous discrete energy levels merge into a band of energies such as the valence band. In semiconductors, there is an energy gap between a valence band and higher conduction band and the Fermi level lies in the forbidden gap. In metals, there is no gap and electrons can move freely (conduct). An insulator means having a large gap (no free conducting electrons). With temperature added, thermal energy can excite electrons in a band and the Fermi level can be set at an average occupancy of 0.5 . So some semiconductor electrons can jump up to the conduction band leaving holes in the valence band. Near absolute zero, electrons fill to the Fermi level with a number of sub-bands below it depending on the applied B field.

In IQHE, increasing the B field increases the degeneracy of each LL. That means that the Fermi level will fall with increasing B field. When the Fermi level lies between Landau energy levels, then all lower Landau levels will be filled. Or we could say that a decreasing B implies that each LL holds fewer electrons and the Fermi energy will go up. “But rather than jumping up to the next Landau level, we now begin to populate the localized states. Since these states can’t contribute to the current, the conductivity doesn’t change. This leads to exactly the kind of plateaux that are observed with constant conductivities over a range of magnetic field” [9]. There is a strange conspiracy that the “current carried by the extended states increases to compensate for the lack of current transported by the localized states. This ensures that the resistivity remains quantized…” [9].

Edge Potential and currents:

Circular motion of electrons is geometrically blocked at the side edges of a thin Hall strip. Essentially, the electron performs half a circle there, bounces back and executes another sequential half circle. This is called “skipping motion” in which electrons can only move in one direction and cannot backscatter from impurities. The net result is a dissipationless edge current flowing forward on one side and flowing backwards on the opposite side [1] (chiral motion). This persistent circulating current is real and measurable. Potential V(x) is highest at these edges, and the edge material acts as a metal. The Landau levels are pushed up at the edges and can rise above the Fermi level. But the bulk in-between is more like an insulator. Impurity scattering is low at these edges, but yet impurities are important for the emergence of the Hall plateaux [1]. The population of edge states traverses the band gap between the valence band and conduction.

On an energy diagram E versus distance across a Hall strip ($0 \leq y \leq W$), each Landau level has a “bathtub” shape (flat on the bottom and rising strongly in energy at the edges). For a given Fermi level, several of these bulk LLs may lie below that level. For example, at plateau i = 2 may have LL n = 0 and n = 1 lying below it. The LL extended states crossing the Fermi energy level correspond to the transitions between plateaus (the “edge states”). Some sources suggest that direct current may be “carried entirely by the edge states.” With high B fields, the electrons that carry current are confined to the edges by the Lorentz force, one for each LL.

When a y- potential difference is introduced across the width, more electrons are introduced across the width and accumulate more on one side than the other -- the bathtub is tilted towards one side. The fermi potential is the same on both sides. Hall voltage gives the Hall conductivity $\sigma_{xy} = I_x/V_H = e^2/h$ [1] (and the appearance that current is carried by the edge states). So, the bulk of the electron gas is an insulator, but along its edge, electrons circulate as an example of the quantization of Berry’s phases [22]. This is related to the concept of “topological insulators” with conducting edge states where “spins of opposite sign counter-propagate along the edges.” (quantum spin Hall [QSH] states)

The most important observable in IQHE is that cross-conductivity is quantized. But if a cross voltage has been built up at equilibrium, why should there still be any current? The answer is that there is always current at the edges of the Hall width, and current in-between can flow from edge to edge. That flow may be incremental widthwise from one LL state to a neighbor and then on to an edge.

Integer Quantum Hall Effect:

The Quantum Hall state is the simplest example of a topologically ordered state and occurs for an electron gas in two dimensions. The Hall conductivity changes stepwise with increasing magnetic field. But, for ultra thin and ultra cold samples, the physics becomes quantum mechanical and crosswise Hall conductance can change by integer steps! $\sigma_{xy} = \nu\, e^2/h$. This is the Integer Quantum Hall Effect (IQHE). The steps or plateaus have incredibly precise values enabling ultra-fine electrical measurements.
von Klitzing [2] in 1980 was the first to discover that conductivity here was exactly quantized and won a Nobel prize in 1985) [again see Figure 1].

A big question is “Why do the steps change by integer multiples?” and “Why are the plateaus broad?” rather than changing with magnetic field like the Hall formula. Thouless helped provide answers to these questions. The plateaus are broad and stable due to Anderson localization between quantized Landau energies. These plateaus exist “when the Fermi energy crosses an extended state level.” Why the conductance changes by integer multiples is given by advanced topology arguments utilizing Chern theory such as that in the TKNN formula. The IQHE conductance is robust because it is a topological invariant of the system immune to deformations [9]. A plateau means that the delocalized sub-bands are completely filled. The conduction electrons cannot jump from one energy level to another, since there are no available energy levels for them. As a result, the scattering of conduction electrons, with loss of energy, cannot happen.” [17]

Attempts to model Quantum Hall transitions included an early use of semi-classical percolation and quantum tunneling. This is still sometimes used but no longer stressed. Delocalization may now be discussed using Topological Field Theory [wikipedia]. There is something mysterious about half-filled Landau levels that makes them special and suddenly metallic. No theory fully explains why the quantization is so perfect and unaffected by the geometry and purity of the material [21].

“Laughlin Gauge Argument”:

Most explanations of Hall quantization are advanced and difficult. The first explanation is the simplest and most referenced [13] -- but still tricky. In 1981, Laughlin considered a 2D rectangular metal strip of length L and width W bent into a circle and also having a normal magnetic field Ho everywhere on the loop (e.g., from an imaginary magnetic monopole). He considers the “disordered case with the Fermi level in a mobility gap..” Let there be a current I resulting in voltage drop V across the width by the Lorentz force. He then considers what happens when magnetic flux is introduced down through the middle of the circle (where magnetic flux is defined as the field times the cross sectional area). For this we need to first look at the Aharonov-Bohm (AB) effect of the vector potential A on electron phase. A is important because of canonical momentum in the Hamiltonian: $H = (p - eA)^2/2m + eE_o y$. There is a field B inside the solenoid of radius R but no magnetic field outside, just a vector potential field. For this A field around the outside of a solenoid (or uniform A field around the ring in this case) at radius rho:

$A_{out} = B_o R^2/2\varrho = flux/circumference = \varphi/L$, Then the AB phase change will be:

$eA\Delta x/\hbar = eAL/\hbar = e\varphi/\hbar$.

If we insist that the phase around the ring be a single valued function (rather than a multivalued winding function) then the total circle phase change must be integer multiples of 2 pi. So,

$AB\ phase = n2\pi = 2\pi eAL/h,\ or\ A = nh/eL\ for\ extended\ states\ (or\ n = ALe/h = \varphi\, e/h)$.

Now add one magnetic flux quantum, $\Phi = h/e$ h/e (so delta n=1). Laughlin says that this sort of gauge invariance requirement maps the system back into itself.

This is an interesting result, that one magnetic flux quantum changes the AB phase by one wavelength around L.

Now notice that power = dU/dt = VI, but $V_x = \oint_L E_x dx = -d\varphi/dt,\ so\ I = dU/d\varphi = dU/LdA$.

Laughlin then claims that one electron per LL is transferred from one edge of the strip to the other edge by ratcheting in successive stages across the width (shift register). This shifting is related to the magnetic lengths and y's discussed above under "Landau Levels" . Current flow in the x direction drives a voltage in the y direction. This current is the transfer of n electrons across the width so that

$I_y = ne/\Delta t = neV_y/\Delta\varphi = ne^2/h$! (using the dt from Faraday's law above).

So, Hall current in the y direction is quantized.

[Of course, there are some assumptions and details left out and still to be addressed, as they are in references [12] [13]]. He adds, "At the edges of the ribbon, the effective gap collapses and communication between the extended states and the local Fermi level is reestablished."

Many articles present the above argument as a "Corbino Annulus" instead of a ring. This model originated in a 1911 study on magnetoresistance. Insertion of central flux then causes migration of charge from the inside radius to the outside.

Fractional Quantum Hall Effect (FQH).

Beyond the Integer QHE: In 1982, Stormer and Tsui first discovered a new quantum Hall effect showing that the ratio of electrons to magnetic flux quanta can occur in p/q integers like ⅓ or ⅖ ! Particles can act as if they had a fraction of the charge on the electron. This is a new state of matter. Remember from above that the IQHE identified one electron state to a Landau level and a magnetic flux quanta. In general, the microscopic origin of the FQH remains unknown, a big work in progress. But Laughlin presented reasoning for the special case of a 1/q state and eventually won a Nobel prize (along with Tsui and Stӧrmer). The FQH requires a "many-electron wave function" (like the 1983 Laughlin example) resulting in fractionally charged "quasiparticles." This is a type of Bose-Einstein condensate in which electrons are bound with an odd number of vortices which can have neighboring depleted charge regions leading to effectively fractional charge.

Resulting composites may be "anyons" that are neither fermions nor bosons. This is dominant in FQE theory, but no anyon has been conclusively seen experimentally. If they do indeed exist, the FQH is the place to find them. The IQHE depends on absence of electron-to-electron interaction, but the FQH depends on it and wants smoother surfaces. The vast number of fractional FQE bands currently requires doing experiment first and trying to formulate theory patterns second. IQHE and FQH are examples of emergent collective order supposedly not deducible from fundamental physics but only from experiment. This follows the new philosophy of Philip Anderson's "More is Different" and Robert Laughlin's "The end of reductionism." The FQH phenomenon are very similar to IQHE except for the transfer of fractional quantum numbers.

FQE is an example of "topological order" with patterns of long-range entanglements, and the changing from pattern to pattern requires a phase transition. This concept lies beyond that of topological insulators, topological superconductors, and traditional Landau symmetry breaking. It may also include high temperature superconductivity and also the IQHE above with a "Chern number of the filled energy band." FQE has Chern-Simons gauge theories as their effective low energy theory. Topological order has "quantized non-Abelian geometric phases of degenerate ground states." (Wikepedia).

For the IQHE, we depend on material disorder. But FQE needs minimal disorder (cleaner samples) to show its fractional value plateaus.

Kosterlitz-Thouless (KT) Transition: Earlier Work.

Before 1960, it was believed that long range order in two dimensional solids was impossible. In the 1970's, a new "topological order" was discovered in which 2D vortices and anti-vortices (which are not whirlpools) pair together allowing unexpected 2D superfluidity and superconductivity. A 1972 "KT" paper was titled, "Long range order and metastability in two dimensional solids and superfluids" [15]. The authors first considered standard dislocation theory and the pairing of "up and down" dislocations but noticed that their observations should also pertain to vortices in superfluids as well. At low temperatures, pairs of "opposite" dislocations pair up closely, but at high temperatures they freely separate and allow a viscous response. They studied what is called the XY model (2D classical rotor or spin model) on a 2D lattice. The KT transition lies between high temperature direction correlations (which decay exponentially fast) and power-law low temperature decay. A Russian, Vadim Berezinskii, did similar work resulting in the name "BKT transition." It was noted that superfluid vortices can form above a critical temperature but not below it. Or, vortices and anti-vortices are free above a critical temperature but paired very close below it. This is a collective phase field unbinding effect that is universal in variables regardless of the chosen system being studied and correlation lengths diverge exponentially [15]. Again, renormalization group equations seem to apply.

A KT transition has been confirmed experimentally in proximity-coupled Josephson junction arrays, and "quasi-long range order" has been applied to thin films of superfluid helium, thin-film superconductors, and other systems.

Duncan Haldane:

Duncan Haldane is a British physicist who did his initial work on one-dimensional chains, and 1D seems less glamorous than the 2D electron gas problems discussed above. In 1981, Duncan Haldane realized that he could apply KT ideas "to the quantum mechanical 1D spin chain if he turned one of the spatial dimensions into time. Then the vortices of KT would become tunneling events between different topological states." [19].

In 1986, neutron scattering was applied to a mixture CsNiCl which has magnetic 1D chains making it a quasi-1D compound and verified some of Haldane's theories. He later discovered many interesting and unexpected new properties [17] which contributed to later advances in condensed matter physics and also had similarities to the 2D physics. Haldane was the youngest of the three winners (b 1951) and had studied under Philip Anderson. Examples of his 1D problems include chains of magnetic atoms, large spin Heisenberg anti-ferromagnet, chains of fermions versus bosons, 1D conductors (quantum wires and now carbon nanotubes), and 1D electron gas. His 1982 paper on spin chains showed topological properties due to "the collective action of the whole chain." There are "topologically protected excitations that behave like Majorana fermions, which are their own antiparticle." He has also been contributing to the understanding of the fractional quantum Hall effect (FQE). Advanced topological topics being used include: Chern Simons theory, O(3) non-linear sigma model, solitons, and instantons. And like the previous discussion, there are analogies of these solid state concepts in high energy physics. For example, Laughlin believes that the quark charges of ⅓ and ⅔ e may have an origin similar to that of the effectively fractional electron charges in the FQE.

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