Propagators

Dave Peterson, 11/8/23-12/26/23, Revised 1/31/24.

Subject: Particle Interaction Math for Quantum Field Theory.

In Feynman diagrams, and usually pictured near the middle of them, momentum space "propagators" represent the virtual particle inner-lines participating in an evolution from one input state to another {see picture example in Figure **A** below}. They are presented as mathematical expressions that enable propagation calculations of probability amplitudes from particle scattering initial states to final states across space-time as Kernels in integrations -- *thus acting like "Green's functions."* Feynman "internal line" propagators are "free field" terms with the ideal absence of forces or current sources, J, from space-time location "x" to y : $D(x\text{-}y) = Z_o \equiv Z[J{=}0]$. {For a visual picture of a free field propagator function, see Figure **K(x,t)** below}. Propagators facilitate momentum transfer between sources in Feynman diagrams and can be key to mathematical generation of higher order Green's functions for calculating perturbation terms for particle interactions. They are essentially "S-matrix" elements of unitary evolution operators $U(t, t_o)$ for advancing wave-functions or fields.

A Feynman "path integral" is an expression for a propagator over all possible paths in configuration space and constitutes an alternate formulation to Schrodinger theory and quantum field theory but one based on Lagrangians rather than Hamiltonians. "The path integral has racked up so many successes that many physicists believe it to be a direct window into the heart of reality." "It's how the world really is [Wikipedia]" But, it "is also more of a philosophy than a rigorous recipe," and applying it can require ingenuity.

The discussions here are still on the topic of "what quantum field theory is about" *{continuing [dpQFT]}* rather than focusing on detailed mathematical techniques for doing calculations. However, since the subject is mathematical, some math forms still have to be mentioned and inter-related. The primary arena is quantum mechanical "amplitudes" prior to forming real particle scattering probabilities; and these amplitudes live in complex or even hypercomplex spaces. That means that ordinary intuitive and pictorial understandings might not suffice.

Utilizing Quantum Field Theory:

A major **goal** of quantum field theory is the calculation of relativistic scattering amplitudes and experimental probabilities often stated as interaction area "cross sections" σ and "differential cross sections" dσ/dΩ for particle scattering experiments. *Total* $\sigma = \int_o^{4\pi} (d\sigma/d\Omega)d\Omega$ where Ω is external solid angle at observations. Going back one level in depth, $d\sigma/d\Omega = |\mathcal{M}|^2/64\pi^2E^2_{cm}$ where $\boldsymbol{\mathcal{M}}$ is a commonly used "invariant amplitude" or "scattering amplitude" in complex space. Particle decay rates also have "width" $\Gamma \propto |\mathcal{M}|^2$. A related concept is the historically important **"scattering matrix" S** and the "transition matrix" T with operators $\hat{S} = 1+i\hat{T}$; and these can be expressed in terms of $\mathcal{M}$. The transition matrix element is from an initial to a final state ψ is $T_{fi} = -i \int d^4x\, \psi^\dagger_f(x)V(x)\,\psi_i(x)$.

The S-matrix is an operator mapping free particle in-states to free particle out-states, and Feynman diagrams are useful in guiding the calculation of S-matrix perturbation terms. As an alternative to "Canonical quantum field theory," Feynman's "path integral formalism" of quantum field theory "represents transition amplitudes as a weighted sum of all possible histories or pathways of the system from the initial to the final state in terms of either particles or fields. It was Dyson who in 1948 realized that "the Feynman theory is essentially nothing more than a method of calculating the S-matrix." The transition amplitudes are effectively elements of the *S*-matrix, and the Feynman method uses a set of now well-known "Feynman rules" for the calculation of the S-matrix. Dyson was able to derive the Feynman method from the canonical quantum field theory {e.g., Schwinger-Tomonaga formalism of the 1940's} thus proving their equivalence.

The Feynman path integral formalism provides one way to calculate transition propagators such as **K**$(x, t; x_0, t_0) = \langle x|U(t, t_0)|x_0\rangle$ telling how states evolve between initial space-time states at "0" to a future time t and position x using a unitary evolution operator U *(and reading symbols from right to left).* The symbol "K" stands for an integration-**k**ernel propagator "amplitude" *{behaving as a "Green's function" …discussed below}.*

Propagators aren't just for use in QFT but may also be applied to ordinary quantum mechanics [e.g., Liboff p 161] . A simplest case may be for a Bell shaped *{or "Gaussian"}* wave packet evolving and spreading out freely with constant group-velocity {momentum $p_o = \hbar k_o$ and hence constant kinetic energy KE = $p_o^2/2m$; and "freely" means without forces and no varying potential energy over space and time}. [See discussion in the Appendix at end below].

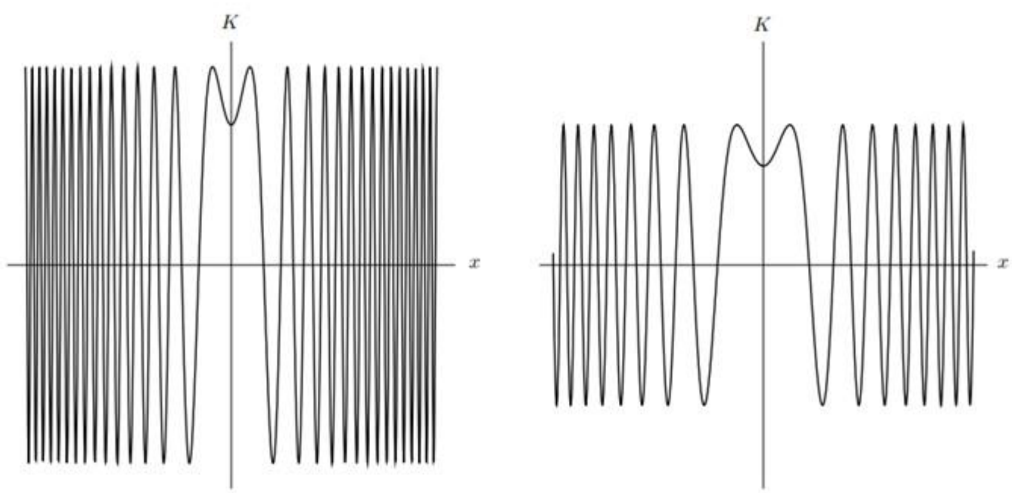


Fig. 1. Real part of the free particle propagator at a positive time, with $x_0 = 0$.

Fig. 2. Same as Fig. 1, but at a later time.

Figure K(x,t): Picture example of a Free Particle Propagator Waveform in space x and evolving with time t ≥ 0. Ref: [Berkeley]. Rapid variation of K for x away to the right or left from x_o=0 effectively produces destructive interference.

A conventional way to write K is the "path-integral form:" $K(b,a) = \int_a^b \mathcal{D}x(t)\ e^{iS/\hbar}$ where "prefactor" $\mathcal{D}$ or "integration measure" includes a "normalizer" factor times a product of all the dx's needed for integrations over all possible paths. Normalization means that $\int K(x\text{-}y, \tau)dy = 1$ where T is "proper time" from space-time location a to b. The "S" term here is called "**action**" and may sometimes be considered as total quantum phase accumulated along a path. The name "propagator" derives from formulas such as $\psi(b)=\int K(b,a)\psi(a)d^3x$, where the Green's kernel K enables the transformation from one quantum state at "a" to a later state at "b." We could require that time of events $t_b > t_a$ for propagation towards the future, so $K = K^+$. But propagation backwards in time is also allowed for the "Feynman propagator" in quantum electrodynamics {"QED," $\Delta_F(x,y) = \Delta^+ (t_x>t_y)+ \Delta^- (t_x< t_y)$ }.

From amplitudes, one can calculate probabilities of transition from a state at "a" to a state b: $P(b,a) = |K(b,a)|^2$ which can also be expressed via the S-Matrix as complex "squaring," $P = S_{ba}^* S_{ba}$. For two-particle scattering like that in Figure **A** {below} with incoming Feynman diagram "external lines" particles #1&2→ 3&4 outgoing asymptotic lines might use a kernel labeled "K(3,4:1,2)." There is also the case of one particle in and the same single particle out (such as radiative corrections for anomalous magnetic moment of the electron or the Lamb shift). The path integral is an expression for the propagator in terms of integration over an infinite dimensional space of paths in configuration space either based on position or momentum (which is a more frequent choice).

Feynman path integrals are based on **Lagrangians**, $\mathcal{L}$, which are Lorentz <u>scalars</u> and have the great virtue of ensuring Lorentz invariance, while the alternative "Hamiltonian formulation" must select a particular time parameter and hence lack an obvious invariance. "**Action**" $S = \int L\, dt$ or $\int \mathcal{L} d^4x$, and the path integral is $\mathbf{\Delta} = \int D_x\, e^{iS/\hbar}$ where action acts somewhat like a complex phase and "D" is a weighing factor over possible paths. Label "L" is used for classical KE-PE = T -V type energies, and label "$\mathcal{L}$" for densities over space-time. Note that Lagrangian terms may be added together. In quantum-electrodynamics (QED) we can have:

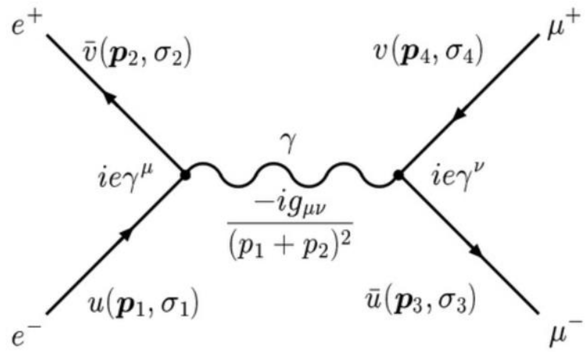
$\mathcal{L}_{qed} = \mathcal{L}_{EM}(A) + \mathcal{L}_{Dirac}(\psi) + \mathcal{L}_{interaction}(jA)$. And then actions will add too: $S_{qed} = S_{em} + S_{Dir} + S_{int}$. The "Standard Model" begins with a long list of Lagrangians all added together to give $\mathcal{L}_{SM}$.

If an interaction source, usually labeled "J," is present, then the path integral is labeled $\Delta[J] = N\int D_\phi \cdot e^{iS[\phi,J]/\hbar}$ {label ϕ for "field"}. The author *Zee ["NUT"] calls this path integral "Z[J]" along with another useful "generating functional" W[J]. The symbol "Z" comes from an analogy to the <u>"partition function"</u> {the first topic on page one of Feynman's book on Statistical Mechanics}. But, Z and W are mainly used for multiple fields for higher order "Green's functions" for Feynman graphs – not too useful for the simpler introductory discussions here for only a "handful" of particles. <u>See Notes at end</u>}.* $Z_o = Z[J{=}0]$ <u>as a propagator</u> is for free motion without potentials or current sources.

The details of path integrals or "sum over histories" will not be discussed here but are given in many textbooks. They often depend on convenient "Gaussian integrals" for Lagrangians having "quadratic forms," e.g., like "Klein-Gordon" $\mathcal{L}_{KG}$ {mentioned below}. "ax^2+bx+c" is a quadratic form; and recall from statistics that the whole area under a Bell curve is a simple $\int \exp[-ax^2/2]dx = (2\pi/a)^{1/2}$.

One of the first tasks of Path Integrals was successfully <u>deriving</u> the Schrodinger equation in a manner similar to that of propagating Huygen's wavelets. Unfortunately, "Feynman's time-sliced approximation does not exist for the most important quantum-mechanical path integrals of atoms, due to the singularity of the Coulomb potential at the origin." This problem was eventually solved by Hagen Kleinert using a clever trick {I knew him in college – a dynamic marvel at CU who then thrived in Berlin}.

Again, calculations of particle scatterings may be facilitated as products of piecewise "Feynman rules." For the special portion of particle-interaction-vertices, these can be read off from the details of interaction Lagrangians by removing "external factors." So an electromagnetic interaction $\mathcal{L}_{int} = ie\bar{\psi}\gamma^\mu\psi A_\mu$ results in an ***"ie γ^μ"*** {gamma-matrix times charge} being attached as a rule to every QED vertex in the Feynman diagram (as in Figure <u>**A**</u>). The rules for virtual <u>internal lines</u> are the various types of propagators whose calculations might be "non-trivial" and dependent on choice of gauge. But gauge selections have no effect on the final S-matrix elements.



<u>Figure A</u>: <u>Sample Feynman diagram</u> for an "annihilation" $e^-e^+ \rightarrow \gamma \rightarrow \mu^-\mu^+$ with time evolution here from left to right {→ t}. The photon, "γ", is labeled by its propagator Green's function G(k) ∝ $-i/k^2_{total}$ {where k *or p or q* is momentum transfered}. The exterior "legs" come in and go out to asymptotic infinity and hence must obey conservation laws for energy/momentum p's and particle spins, σ's. "Spinor" legs labeled "v" are for anti-matter particles versus spinor-u for ordinary fermions. The diagram "appears" to be in space-time; but particle labels are in terms of momenta; and the whole diagram serves as a guide to perturbation calculations. [Figure source: Quora]

{Note that this picture is an over-simplification. Even though it shows a "vertex" and a single photon exchange, the Coulomb force is felt along a broad path with continual virtual photon exchange. So, "the diagram for exchange of a single photon actually stands for all possible cases of exchange of a single photon. That is, early exchange, middle exchange, and late exchange" and all possible ways of doing it. " Feynman diagrams are actually in momentum space. So, a single diagram is for all possible momenta for the photon" [Mike Jones]. QED calculations make all that work out}.

In addition to classical field theory and QED, quantum field theory Lagrangians also apply to the Higgs field, Yang-Mills theory [e.g., Veltman], standard model (SM), spontaneous symmetry breakings, gauge theory, electroweak theory (EW), and quantum chromodynamics (or "QCD). We might also add: supersymmetry, string theory, topological field theory and perturbative quantum gravity-- if these theories turn out to be "real." With Lagrangians, one also has Actions, S, that can be used in path-integrals; and then relevant propagators may be derived. Path integrals are also useful in non-perturbative "Lattice-QCD" calculations [Lee]. Path integrals allow us "to understand phenomena that cannot be described in the usual perturbative canonical approach [L&B]."

<u>Propagators</u>:

Propagators propagate wave functions {often labeled ψ or ϕ } through time and could be viewed as position-space matrix elements of the unitary time evolution operator $U(t,t_o)$ – that is: $K(x, t; x_0, t_0) = \langle x|U(t, t_0)|x_0\rangle$ -- and often for time $t > t_o$. Convenient lists of Feynman propagators are shown in tables of "Feynman rules" for amplitude transitions. These rules for time-dependent perturbation theory can be calculated in either position space <u>or</u> momentum space and <u>either</u> from canonical Hamiltonian QFT <u>or</u> Lagrangian QFT or by "Path Integrals." Momentum space Feynman rules have the advantage that one can write out invariant amplitudes from pictures (Feynman graphs). Propagators also apply to usual nonrelativistic quantum mechanics scattering and to relativistic QM too. Going back and forth between space-time and momentum space is accomplished using Fourier transforms {"FT", as in G(x,y)=FT of G(p) or vice versa}.

Recall that 3-dimensional FT: $F[f(x)] \equiv f(k) = \int^{\infty}_{-\infty} f(x)\exp[\pm ik\cdot x]\, d^3x$ {with convention choices for normalizer and ± signs}

Moving into quantum field theory (QFT), we care about how fields evolve: one expression is for probability amplitude = $\langle \phi(\bar{x}), t | \phi_0(\bar{x}), t_0 \rangle = \int \mathcal{D}_\phi\, e^{iS/\hbar}$ where $\mathcal{D}_\phi$ is a path integral measure or weight. Propagators describing evolution in "momentum-space" emphasize energy/momentum, $p = p^\mu = (E/c, \vec{p}) = (p^o, p^i)$. {*Note that since $p=\hbar k$ for "wavenumber" and often choosing $c \equiv 1$ and $\hbar \equiv 1$ "natural" units, we have 4-vectors "p"= "k" = $p^\mu = k^\mu = (\omega/c, \vec{k}) = (k^o, k^i)$. Label "q" is also often used for momentum transfer. All of these equivalent names are used below*}.

A common propagator found in tables is the momentum space Green function of the appropriate free particle wave equation [Ait p 161, without interactions]. The Green's function game for a given differential equation is to solve it by first looking for an impulse solution δ or "spike." If differential operator $\hat{L}\, y = f$, find an inverse operator so that $y = L^{-1} f$ by first solving the simpler equation $\hat{L}\, G(x|t) = \delta(x-t)$. Then $y(x) = \int G(x|t') \times f(t')dt'$. *The Green's function G(x,t') is referred to as the kernel of the integral operator* L^{-1}.

A simplest example is Newton's law F = ma where $\hat{L}\, x = \ddot{x} = a = F/m$. We find Green's kernel so that $d^2G(t|t')/dt^2 = \delta(t-t')$ with t' an arbitrary starting time. The solution is $G = (F/m)(t-t')$ so that $x(t) = \int^t_o G(t|t')dt' = \int^t_o (t-t')dt' F(t')/m$. If acceleration "a" = a constant, then the solution is $x(t) = at^2/2$ (and this approach skips an intermediate velocity step).

The Feynman game requires finding such an inverse L^{-1} for quantum amplitude solutions with a preference of transferring this to momentum space first {i.e., express δ by its Fourier transform in k-space and then solve for G}. That is, Feynman in 1949 showed that propagators based on "action" over space-time are simplified if they switch to Fourier transformed momentum space instead. Then integration of Kernels over $d^3x dt$ become d^4k or $d^4p = dE\cdot d^3p$.

A relevant example is a **Klein-Gordon scalar boson** equation $(\partial^2/\partial(ct)^2 - \nabla^2 + (mc/\hbar)^2)\phi = 0$ with applications to spin-zero particles such as the pion, kaon, and pure Higgs boson. If sources are appended on the right side, the equation may be shortened to the symbolic form: $(\Box + m^2)\boldsymbol{\phi} = -V\phi = -j$ which "inverts" to $\phi = j/(p^2-m^2)$ leading to a Feynman propagator rule contribution of factor

$K(p) = \; i/(p^2-m^2)$. Zee [nut] shows that exchanges of KG particles between two sources leads to a net attraction [such as nuclear binding]. *The Lagrangian for KG fields is $\mathcal{L}_{KG}(\phi) = (\tfrac{1}{2})[(\partial\phi)^2 - m^2\phi^2]$ which goes into action $S = \int \mathcal{L}\, d^4x \rightarrow (\int$ by parts$) \rightarrow$ functional integral form $-(\tfrac{1}{2})\int \phi(\partial^2+m^2)\phi$.*

{As a caution, the signs on these equations depend on the signs of the space-time metric, $g_{\mu\nu}$ – a 50-50% choice. Here we have $g_{oo} = \eta_{oo} = +1$, but many (like Weinberg) choose -1. [Steven Weinberg, The Quantum Theory of Fields, Vol. 1, 1995, Cambridge, p 259: "The Feynman Rules"]. $\eta_{oo} = -1$, so $\Box = \nabla^2 - \partial_t{}^2$; momentum $q^2 = -E^2 + p^2$. KG: $\partial^2+m^2 \rightarrow -\partial^2+m^2 \rightarrow k^2+m^2$ in propagator denominator. His momentum space internal line Feynman rule is $\propto 1/(q^2+m^2-i\epsilon)$ }.

{To understand this "inverting process," first solve $(\partial^2+m^2)K(x-y) = -\delta^{(4)}(x-y) \equiv \int -[d^4k/(2\pi)^4]e^{ik(x-y)}$ [in terms of momentum, k, [A. Zee, NUT p 23]] }. For plane-wave propagation, $\phi_p(x)^*\phi_p(y) = e^{-ikx}e^{+iky} = e^{-ik.(x-y)}$.

This expression for δ might be more familiar for the case of FTs in one-dimension where $\delta(x) = (1/2\pi)\int e^{ikx}\, dk$ (and, as every electrical engineer knows, a "spike" impulse function can be viewed as the superposition of an infinite number of plane waves). In QFT, progress along a path uses plane waves along the path, $e^{ik.x}$ or $e^{i(kx-\omega t)}$. The KG operator on this is $(\partial^2+m^2)e^{ikx} = (\underline{-k^2+m^2})\, e^{ikx}$ and it is this that gets inverted for G (or "D" or "K" or Δ or Π – symbols for propagators vary in QFT literature). A shorthand symbolic phrase is: the FT of $\hat{L}\, G_{(KG)} = \delta$ is $(-p^2+m^2)G = 1$, so flip for G. We then have to integrate using the Feynman kernel, and it has a pole in the complex plane (or real energy axis) that requires special thought. For the simpler case of massless photons, m=0), d'Alembertian $\Box e^{ikx} = -k^2 e^{ikx}$; so the propagator $\propto 1/k^2$ (as in Figure A) .

Fermion problems are a bit different: For an internal electron propagator, using the electromagnetic momentum shift, the operator $\partial/\partial x_\mu \equiv \partial^\mu \rightarrow \partial^\mu - ieA^\mu$, the free **Dirac equation is** $\mathbf{(i\partial^\mu - m)\psi = -eA\psi}$ (where single 4-vectors ∂ and A, p or q are all now in Feynman " / slash" notation to include multiplication by γ_μ [Aitkinson, p. 197] where gammas are 4× 4 square matrices and ψ (or u or v) are now the special row or column matrices called "**spinors**." Their rest-frame elements of 1's or 0's for spin-up or down get "boosted" by relevant Lorentz speed transformations to now include momenta and energy weighings (e.g., a weight p/(E+m)). Performing calculations requires multiplying row, column and square matrices together followed by summations and "Trace" theorems (it gets very complex). The propagator here is similar to an inverse of the Dirac operator ∼ 1/(q – m) and is written in the form $\Delta(q) = (q+m)/(q^2-m^2)$ {with numerator q overlayed with a forward-slash / } . Also note that electrons are excitations of a Fermi field but are not quantized by representing it as a set of harmonic oscillators as for BOSE fields due to different statistics and "Grassman numbers" [Hibbs].

As a real example of this, the Compton effect: $\gamma(k)+e^-(p) \rightarrow e^* \rightarrow \gamma(k')+e^-(p')$. As in Figure **A**, there are incoming and outgoing "legs" of a Feynman diagram; but, we include an intermediate single excited electron line, e*, with 4-momentum **q= p+k**. The Feynman rule term for this now includes the term $\Delta(q)= (p\pm k+m)/[(p\pm k)^2 - m^2]$.

As mentioned earlier, the **Feynman rules** apply to parts of the scattering process Feynman diagram, and all parts are multiplied together. As an example of its use, for the total invariant amplitude, $\mathcal{M}$ for the Compton process, we look at a product like:

{Equation "$\mathcal{M}$" and "JDJ" form}:

$\mathcal{M} \propto$ (out-going e-leg wavefunction)× (γ-polarization)× (photon vertex term)×(interior propagator, D) × (other photon vertex term)× (incoming photon polarization)× incoming electron wavefunction u(p)

with momentum p [e.g., Halzen p 142] . And then, forming the actual transition amplitude $T_{i\rightarrow j} = T_{ij}$ requires insisting that energy/momentum be physically conserved as expressed in a delta-function $\delta^{(4)}$(over all in or out momenta/energy along with possible normalization factors, N). In the interaction region, we didn't much care about conservation laws yet. That is, the interactions may be "off-shell," but detectors far away experience "on-shell" conservation. Green's functions are unphysical. They describe virtual (off-shell) particles where $p_\mu p^\mu \neq m^2$ (and we integrate over all p and E). $p^2=m^2$ is "on-shell" {i.e., $E^2 - (pc)^2 = (mc^2)^2$ }. The ingredients of the delta function vary with different observer frames of reference. Weinberg refers to $\mathcal{M}$ as a "delta-function free" transition amplitude.

Notice that in **Equation "$\mathcal{M}$,"** the incoming particle legs carry momentum, $\vec{p} = \gamma m\vec{v}$, and the vertices carries charge (the coupling constant ∼ e) with the "propagator" sandwiched in the middle. This is like $(ev)D_F(ev)$ ~ $\mathbf{J\, D_F\, J}$ like the currents in eqn. **W[J]** in notes at end. We will frequently encounter this important "JDJ form." For Moller Scattering with a photon internal line, Halzen labels this as Transition amplitude $T_{fi}^{(2)} = -i\int j_{\mu\,(1)}(x)(-1/q^2)j^{\mu\,(2)}(x)d^4x$ (JDJ form) and says that when we have two interaction vertices (as in Eqn $\mathcal{M}$), the result must be the 2nd order term in the perturbation series, $T_{fi}^{(2)}$. That is,

$T_{fi} = -i\int j_\mu A^\mu d^4x$ where $A^\mu = -j^\mu_{(2)}/q^2$ as solution of $\Box A^\mu = j^\mu_{(2)}$.

Another common form of propagator is "the Wightman function" or "vacuum expectation value of fields in a time fixed order = $D(x\text{-}y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int d^3p/[(2\pi)^3 \cdot 2\omega_p] \times e^{-ip(x-y)} = \int [d^4p/(2\pi)^4] K(p)\, e^{-ip(x-y)}$. (where $\omega_p = E_p = \sqrt{(\vec{p}^{\,2}+m^2)}$, "on-shell"). *D(x-y) = FT of K(p)* and "provides an amplitude associated with a disturbance in the field ϕ traveling from spacetime location x to y." By definition, this function is also a correlation function or "correlator" used as building blocks of other propagators.

Recall that in statistics, **correlation** is defined as the degree to which a pair of variables are linearly related: $\rho_{x,y} = \text{corr}(X,Y) = \text{covariance}(X,Y)/\sigma_x\sigma_y = E[(X-\bar{X})(Y-\bar{Y})]/\sigma_x\sigma_y$ where E[ ] means expectation value or ⟨ _ _ ⟩ and bar means mean or average value.

In its momentum form, K(p) = The relativistic free particle "propagator for internal lines" $=G_F(p) = \Delta(p) = i/[p^2-m^2+i\epsilon]$ = FT of G(x,y) . The $d^4p = dp^0dp^3 = d\omega dp^3$, and the $i\epsilon$ offset in the complex plane avoids a pole blowup in the integral at $\omega=\omega_p$ ["on-shell," see Evans]. "$i\epsilon$" is a "regulator of the path integral and also defines the 'flow of time.' For unstable particles, the "$i\epsilon$" → $im\Gamma$ with decay width Gamma. The Feynman propagator two-point or two-leg function $D_F(x\text{-}y) = G_2(x_1,x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ describes the vacuum expectation values (VEVs) of two time ordered field-operator products.

{Note: The $p^o = \omega = E$ energy part of integration is done using the Cauchy integral theorem: $\int_c f(z)dz = 2\pi i \sum R_i$ where an integrand is put into the form $f(z) = R_i/(z-z_i)$. And near the z_i pole, "residue" $R_i = \lim_{z\to z_i} (z-z_i)f(z)$. (So Feynman's integral will pick up a factor of $2\pi i$)}.

Beyond QED Propagators:

Rather than the propagator $iD_{\mu\nu} = ig^{\mu\nu}/k^2$, a more general photon propagator can include a choice of electromagnetic gauge, ξ, as given by [Zee, p 141]: $iD_{\mu\nu} = (i/k^2)[(1-\xi)k_\mu k_\nu/k^2 - g_{\mu\nu}]$. **Eqn. γ**

The usual "choice of ξ = 1 is known as the 'Feynman gauge,' and the choice of ξ = 0 is known as the Landau gauge." As usual, the end result must not depend on the choice of gauge.

For **Gluons**, we also use this Eqn. gamma formula (since they are also massless, see list of Feynman rules for "tree graphs" in [Aitkinson, p 544] .

For massive spin-one vector bosons (like the **weak W boson** or $W^\pm$ and Z^o and the heavy Proca photon possibility), we have a boson propagator $D = (i/[k^2-\mu^2])(k_\mu k_\nu/\mu^2 - g_{\mu\nu})$ where μ is effective mass of the particle. This goes along with the addition of an interactive Lagrangian term $-\frac{1}{2}\mu^2 A_\mu A^\nu$ to the usual Dirac Field [Zee p 128][Ait p 341]. In "Fermi Theory" we would say that "the matrix element has the form" $\mathcal{M} = J_\mu^{\text{weak}}\, D\, J_\nu^{\text{weak}}$ (the JDJ form).

For comparison, recall that the propagator for massive spin-0 fields was given by a simpler $D(k) = 1/(k^2-m^2)$.

Propagators are also shown in textbooks for spin 2 gravitons but have tensor terms that are not very intuitive. Zee [nut p 34, 426] shows that their W[J] expressions do lead to attractions.

Appendix:

Coordinate and Momentum Representations:

In the coordinate or "x -representation," operator $\hat{p} = -i\hbar\, \partial/\partial x$, while " $\hat{x}$ operating on a state has the effect of multiplying the state by the scalar x" [Liboff, p851]. So, $\hat{x}$ on an eigenstate of $\hat{x}$ gives $\hat{x}\,|x'\rangle=$

x'|x'⟩. Similarly, p̂ on an eigenstate of p̂ gives p̂|p'⟩ = p'|p'⟩. Ket x' and bra x are orthonormal so that ⟨x'|x⟩ = δ(x-x'). Dirac delta functions are usually used under integrals such that ∫ δ(x-x')dx = 1 or ∫f(x)δ(x-x')dx = f(x'). Technically, δ(x) is a limit of a distribution (such as Gaussian, Sinc, or Lorentzian shape). So, $\delta(x) = \text{Lim}_{n\to\infty}\, n\cdot\exp(-n^2x^2)/\sqrt{\pi}$. As a spike function, it has a Fourier transform form $\delta(x) = (^1/_{2\pi})\int e^{ikx}\,dk$ in momentum or wavenumber space. To inter-relate x and p for the important "transfer matrix" ⟨x|p⟩ , we need a few tricks. One is the "spectral resolution of unity" for continuous x or p: Î = ∫|x⟩⟨x|dx or Î = ∫ |p⟩⟨p|dp ; and this also introduces ∫ that can be used on δ .

So, now, look at $p\langle x|p\rangle = \langle x|\hat{p}|p\rangle = \int \langle x|\hat{p}|x'\rangle\langle x'|p\rangle dx' = -i\hbar\int {}^{\partial}/_{\partial x}\, \delta(x-x')\langle x'|p\rangle\, dx' = -i\hbar\, {}^{\partial}/_{\partial x}\, \langle x|p\rangle$. Then, recalling that dy/y = dℓn(y), we see $\ell n\langle x|p\rangle = ipx/\hbar$, or $\langle x|p\rangle = e^{ipx/\hbar} = e^{ikx}$ {and we could multiply by a normalizer of 1/√h}. These concepts are frequently applied in Fourier transforms and path integrations.

Zee says, "Do you remember that $\langle q|p\rangle = e^{ipq}$? Sure you do. This just says that the momentum eigenstate is a plane wave in the coordinate representation" [Zee NUT p 10] .

Propagator for ordinary quantum mechanics [e.g., evolution from a Bell shaped curve]:

Consider a <u>Gaussian wave packet</u> at time = 0 and variance σ^2 {meaning a Normal probability distribution, $f(x) = (1/[\sigma\sqrt{2\pi}])\exp[-\frac{1}{2}(x-\mu)^2/\sigma^2]$}. But, amplitude is square-root of probability so $\psi(x,0) = \sqrt{f(x)}$ with √ root of the coefficient and half the exponent. <u>The FT of a Gaussian is also a Gaussian,</u> so the initial shape can be given either in coordinate space or momentum space, k. The standard deviation of the Bell curve in momentum space is $\sigma_k = 1/2\sigma_x$; or $\sigma_x\sigma_p = \hbar/2$ – and that <u>is</u> the <u>uncertainty</u> principle for these shapes (but ≥ ½ $\hbar$ for other distribution profiles).

Propagator K = ⟨ x|U|x'⟩ where $U(t) = \exp[-iHt/\hbar]$, and the Hamiltonian is just H = KE = $p^2/2m$. So $K = \int dp\, \exp[-itp^2/2m\hbar]\langle x|p\rangle\langle p|x'\rangle$ -- and then use concepts from the previous paragraph above to finally get:

$$K(x',x|t) = \sqrt{\frac{\tau}{i4\pi\sigma^2 t}}\, exp\left[\frac{i\tau(x-x')^2}{4t\sigma^2}\right] = \sqrt{\frac{m}{i2\pi\hbar t}}\, exp\left[\frac{im(x-x')^2}{2\hbar t}\right],\ \text{with } \tau=2ma^2/\hbar \qquad \textbf{Eqn. Bell Kernel}$$

[Liboff p 162] and [Cal]. And this agrees with Feynman&Hibbs path integral result for the <u>free</u> propagator. [Hibbs,p42]. We could say that the Bell shape results from a superposition of many wavelengths and that the shorter and faster ones keep outpacing the mean flow. The spreading out of the probability distribution over time and distance, P(x,t), is provided in references such as [Liboff] and [Cal].

<u>Canonical QFT</u> {like from Schwinger, Tomonaga, and Dyson} uses standard creation and annihilation operators and Hamiltonians – which have been avoided in the sketch above. For scalar bosons in the interaction picture , we can write a field operator as:

$\phi(x,t) = \int (d^3k/[(2\pi)^3\, 2\omega])\ [\hat{a}(k)e^{-ik.x} + \hat{a}^{\dagger}(k)e^{ikx}]$ creating and destroying plane wave motions [Ait.2nd p 112 for KG field][and <u>Open</u> ref]. Its Hamiltonian is $\hat{H}_{KG} = \int [d^3k/(2\pi)^3 2\omega]\ \hat{a}^{\dagger}(k)\hat{a}(k)\omega$. But, the quanta number counting operator is $\hat{n}(k) = \hat{a}^{\dagger}(k)\hat{a}(k)$, so $\hat{H} \propto \hat{n}(k)$ for non-interacting quantum fields, and these are constants of the motion. Energy is proportional to the number of identical quanta, and the idealization is quanta of little harmonic oscillators. Without interactions, these KG waves will pass through each other like light beams pass freely through each other. For interactions, one could include a ϕ^3 or ϕ^4 or p^3 higher order term in the Lagrangian—<u>or</u> multiple currents J_1, J_2,… . Then, the number of quanta need not be constant.

The essence of "Canonical quantization of fields" is perhaps best described in [L&B p 98+].

For electromagnetic field interactions in QED, we add an interaction Lagrangian term $\mathcal{L}_{int} = -e\bar{\psi}\gamma_\mu\psi A^\mu = -\hat{j}^\mu A_\mu$ as encouraged by local U(1) gauge invariance – j is an electromagnetic 4-current source term, and $\hat{H}_{int} = \int j^\mu A_\mu\, d^3x$. Since there are 3 fields, $\bar{\psi}$, ψ and A, the interaction is similar to having ϕ^3 interacting QFT.

Green's generating functionals "Z(J) and W(J)" of Propagators:

The path integral formulation representation of quantum propagators as sums over classical paths can also be used to compute correlation functions as averages of operators in the "Heisenberg picture." The functional Z[J] is the generator of correlation functions-- the time-ordered Green's functions defined by functional differentiation of Z with respect to the sources J. An important example is the 2-point correlator Green's function $G^{(2)}(x_1-x_2)=-i\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle$ the time ordered product of field operators ϕ, and T is the time ordering operator (using two Heavyside step functions).

"Z[J]" generates all types of Feynman graphs enabling a full perturbation calculation of interactions (often together in a "power" series). It can be stated in several different forms and is coupled with another generator series labeled "W[J]."

$$Z[J] = N\int D_\phi \exp[i\int d^4x[L(x)+J(x)\phi(x)] = N\int D_\phi\, e^{iS[\phi,J]/\hbar} = \exp[-i/2\int d^4x d^4y\, J(x)\Delta_F(x-y)J(y)].$$ **Eqn. Z[J].**

[Kaku p277] $Z[J]=N\int D_\phi\, e^{iS[\phi,J]/\hbar} \propto e^{iW(J)}$ That is, **W(J) is defined by** $\mathbf{Z[J]= Z[J=0]e^{iW(J)}}$. **Eqn. Def W.**

or **W(J) = -iℓn(Z).** The "free energy of interacting two currents" is:

$$U = W(J) = -\tfrac{1}{2}\int\int d^4x\, d^4y\, J_a(x)D(x-y)J_b(y)$$ **Eqn. W[J].**

{a "JDJ" form that **Zee** [ref: "Simply"] labels as $\mathcal{F}(J_1,J_2)$ } where D(x-y) is a space-time propagator between y and x without interactions (and there are similarities to eqn. $\mathcal{M}$ above). If we wish to focus on energies rather than amplitudes, they are contained in the exponent of Eqn. Z[J] and can be brought down by taking the log of Z – i.e., ℓn(Z). **"W"** is like an interaction energy times a time-duration, E·T, and is useful in demonstrating the existence of attraction or repulsion forces (e.g., [Zee NUT]). One of the most interesting and powerful uses is saying that energy $\mathbf{E= -(-1)^S W(J)}$ for particle spin S so that scalar spin S=0 implies negative energy (attraction), photon spin 1 implies positive energy (repulsion), and graviton spin 2 implies negative energy (attraction). {Electron half-spin is a different ball-game and requires both e^- and e^+ together}. Z and W are mainly used for having multiple quantum fields for higher order Feynman graphs – but not too useful for the simpler discussions here.

The **symbol "Z"** comes from a close analogy to the **"partition function"** of statistical mechanics {"**Z**ustandsumme" in German}: "generating functional" $Z = \sum e^{-Ei/kT} = \sum e^{-\beta Ei}$ {where k is Boltzmann's constant, k_B, and β = 1/kT—and there are similarities to the $Z[J]\propto e^{iW[J]}$ above} . The "canonical probability distribution" for state "r" or "sum over states" is $P_r = \exp[-E_r/kT]/Z$.

For any QM-operator for a physical observable, Â, the expected value is $\bar{A} = \sum \langle i|A|i\rangle\, e^{-Ei/kT}/Z$ as a "fundamental law." So, mean energy is $\bar{E} = \sum E_r P_r = -[\sum \partial(e^{-\beta Er})/\partial\beta]/Z = (-\partial Z/\partial\beta)/Z = -\partial\, \ell nZ/\partial\beta$. "Entropy" is identified with $S \equiv k_B(\ell nZ + \beta\bar{E})$ and mean pressure with $\bar{p} = kT\, \partial\ell nZ/\partial V$ [Reif SM p 213

1965 – "all the important physical quantities can be expressed completely in terms of ℓnZ".].
Helmholtz free energy $F \equiv \overline{E} - TS = -kT\ \ell n\ Z$.

The mean variance of the distribution can be shown to be $\sigma^2 = -\partial\overline{E}/\partial\beta = \partial^2 \ell nZ/\partial\beta^2$.

Wikipedia says that "Wick rotation" connects statistical mechanics to quantum mechanics by replacing inverse temperature" $\beta = 1/kT$ with $it/\hbar$. The use of "imaginary time" is like the old Minkowski metric in relativity versus a Euclidean form with all ones on the diagonal. "Goodbye ict" often sees special uses of "hello again" imaginary time.

For QFT, we could write $J = J_1+J_2$ for the case of a disturbance at 1 being absorbed by a sink at location 2. The Z[J] view is useful in the Dyson-Schwinger equation of QFT. It is meant to be a collection of Green's function, a sum of $1+G_1+G_2+G_3$ for one J, 2 J's, 3 J's … Zee prefers symbol G(x-y) for propagation of a particle between y and x in the presence of interactions.

Two particles scattering off each other (sources 1 and 2 → sinks 3 and 4) require finding a term in the Z expansion containing $J(x_1)J(x_2)J(x_3)J(x_4)$ called $G(x_1,x_2,x_3,x_4)$ – a 4-point Greens function with interactions [NUT p 49] .

Zee [NUT p 167] introduces what he calls the "Central Identity of Quantum Field Theory"

$\int D_\phi \exp[-\tfrac{1}{2}\phi\cdot K\cdot\phi - V(\phi)+J\cdot\phi] = e^{-V(\delta/\delta J)} \exp[\tfrac{1}{2} J\cdot K^{-1}\cdot J]$

This is again a "JDJ" form like W(J).

An entity called the "connected correlation function" of operator valued fields [L&B p 198] is stated as the Green's function:

$G_{ij} = \langle \phi_i\ \phi_j\rangle_t = [1/Z(0)]\ \partial^2 Z(J)/\partial J_i \partial J_j\ |_{J=0}$ Or, $G_n = \partial^n/(\partial J)^n Z[J]\,|_{J=0}$. { both Z and Gij can be expressed in Feynman diagrams.} "All Green's functions can be extracted from Z[J] through differentiation" with respect to their currents, J [L&B, p201].

Then see [Zee,p27][Kleiss] {both refer to Z as a path integral – as also does Wikipedia}. Again,

$\mathbf{Z}[J] = Z[J=0] \sum_{n=0}^{\infty} [iW[J]^n]/n! = \sum_{n\geq 0} J^n G_n/n!$ And $\mathbf{W}[J] = \ell nZ[J] = \sum J^n C_n/n!$

References:

[Zee NUT] A. Zee, Quantum Field Theory in a Nutshell, Princeton, 2003, 518 pages. Current Reading along with an easier book, A. Zee, Quantum Field Theory as Simply as Possible, 2023.

STANDARD TEXTBOOKS on quantum field theory (QFT) :

Stephen Weinberg's QFT vol 1 1995; Michio Kaku QFT 1993; David McMahon QFT Demystified; Halzen & Martin Quarks & Leptons 1984, Silvan Schweber: QED and the men who made it, 1994; Bjorken and Drell Relativistic QM 1964; [Ait] Aitkinson and Hey, Gauge Theories in Particle Physics, 1989;
[L&B] Lancaster & Blundell QFT for the gifted Amateur; Oxford, 2014, 485 pages
[Hibbs] Feynman and Hibbs QM and path integrals 1965 {note that the newer Dover edition corrects 879 errors! from the earlier version}.

[dpQFT] DP http://www.sackett.net/**AboutQuantumFieldTheory**.pdf November, 2022. 16 pages.
[Evans] "The Feynman Propagator and Cauchy's Theorem," Tim Evans, 2018.
[UK] https://www.imperial.ac.uk/media/imperial-college/research-centres-and-groups/theoretical-physics/msc/current/qft/handouts/qftfeynmanpropagator.pdf

[Kleiss] Quantum field theory for the electroweak Standard Model, 137 pages, https://cds.cern.ch/record/1143384/files/p1.pdf
[Lee] Frank X. Lee, "Path Integrals in Lattice Quantum Chromodynamics," https://arxiv.org/pdf/0710.4103.pdf {also in Zee NUT 2003}.
[dpAboutQM] http://sackett.net/QuantumMechanicsWithoutMath.pdf. 2/26/22 15 pages.
[Open] Ondrej Certik, Open source theoretical physics book, [Includes 8: Quantum Field Theory and Quantum Mechanics]. https://www.theoretical-physics.com/0.1/index.html e.g., 3.4. Fourier Transform https://www.theoretical-physics.com/dev/math/transforms.html

[dpLearning] "LEARNING QUANTUM MECHANICS AND RELATIVITY," 7/22/15 , in http://www.sackett.net/DP_Stroll2.pdf pages 29-54,
[Veltman] https://www.nobelprize.org/uploads/2018/06/veltman-lecture.pdf

R.P. Feynman, Statistical Mechanics, A Set of Lectures, Benjamin, Inc. 1972
F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw-Hill, 1965.
[Liboff] Richard Liboff, Introductory Quantum Mechanics 4th Ed, Addison-Wesley, 2003, 880 pages.
[Cal] https://sites.astro.caltech.edu/~golwala/ph125ab/ph125_notes_l14.pdf and also 15.pdf too. 2008
[Berkeley] Physics Class, "The Propagator and the Path Integral," 2020, 32 pages.
https://bohr.physics.berkeley.edu/classes/221/1112/notes/pathint.pdf
Other pictures of propagators K(x,t) are shown in VALDEMAR MELIN 2021:
https://www.diva-portal.org/smash/get/diva2:1568326/FULLTEXT01.pdf
[Quanta] https://www.quantamagazine.org/how-our-reality-may-be-a-sum-of-all-possible-realities-20230206/?mc_cid=1f2d3264d6

[Marcos] https://www.marcosmarino.net/uploads/1/3/3/5/133535336/path-integrals.pdf Path integrals in quantum theory Marcos Mari˜no 83 pages

NOTES:

On Attraction: S. Deser How Special Relativity Determines the Signs of the Nonrelativistic Coulomb and Newtonian Forces https://arxiv.org/pdf/gr-qc/0411026.pdf
The bottom line is that particles of even spin mediate attractive forces, particles of odd spin mediate repulsive forces. One has to look a bit more carefully, in order to understand the, apparent, exceptions and how they fit. https://www.physicssayswhat.com/2017/03/26/virtual-attraction/
classical fields are best represented by coherent states in QED. These states are an infinite superposition of photon number states —

"In 2017, a physicist finished a two-decade labor of love, a precise calculation of the electron's g-factor that required computing hairy equations from 891 Feynman diagrams. The result revealed just the fifth term in the series – to a^5. In 1952, Freeman Dyson was the first physicist to appreciate that perturbative quantum theory was probably doomed. Growth in the number of diagrams contributing to the a's will eventually beat the shrinking of the powers of alpha, and the sum will grow untamed toward infinity. Factorial growth would mean that calculating a^9 will require roughly 9!=362,880 diagrams.