Subtle is the Lord: Three topics included here are: General Covariance, Einstein's Light Quanta, and the moving Magnet versus conducting wire loop.

Dave Peterson, 7/26/21-8/6/21

A Question about Einstein's Theory of Gravitation

The discussion here is a bit "technical:" The Cosmo Zoom meeting of 7/19/21 wished a better understanding of the connections between "the principle of General Covariance," the inclusion of the Ricci trace term R (or trace T), and the baffling "Bianchi identities" in the Einstein field equations of general relativity, $G_{\mu\nu} \propto T_{\mu\nu}$ *(the Einstein gravitation tensor is proportional to the "stress-energy tensor" sources).* One factor relating these three concepts is local conservation of energy/momentum in general relativity.

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = -\kappa T_{\mu\nu} \;\Leftrightarrow\; R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{g_{\mu\nu}T}{2}\right), \qquad G^{\nu}_{\mu;\nu} = 0 \;\{Eqns. Pais\}$$

{Abraham Pais Book, p. 256, Pais eqn. 14.15 for 11/25/1915, Bianchi Identity for G from Weyl 1917}

"General covariance" *("GC," or general invariance, or diffeomorphism covariance)* is a gauge symmetry implying that the form of physical laws is invariant under arbitrary differentiable spacetime coordinate transformations. There are no "preferred" frames of reference. "A physical law expressed in a generally covariant fashion takes the same mathematical form in all coordinate systems and has often been expressed in terms of **tensor** fields." "If a tensor equation holds in one coordinate system, it holds in all." Einstein considered GC as a fundamental heuristic principle of his Theory of Gravitation -- *(but, the exact form of GC and its implications have always been in dispute – is it really a necessary and profound principle?).*

The principle of equivalence (PE) was his most basic fundamental principle for developing General Relativity (GR), and PE goes along with using a "symmetric" metric tensor ($g_{\mu\nu} = g_{\nu\mu}$) and symmetric Christoffel connections (Γ's). These in turn imply that GR has no "torsion" *(a simplest example of torsion is a left or right handed helix or an Archimedes Screw).* Another basic principle towards a theory of gravitation is an assumption of conservation of energy/momentum. Instead of this combined name, we might prefer "momentum four-vector" or just symbol **P** = [E/c, **p**] = p_μ , where energy E is just the first component of **P**. Total energy can only be defined for "asymptotically flat" spacetime (like Schwarzschild space).

General Relativity is expressed in the language of Riemannian geometry. Contractions of the 4-index Riemann curvature tensor include the Ricci tensor $R_{\mu\nu}$ and the Curvature scalar $R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}$ *(a summation along the diagonal of Ricci – a "**trace**"). {For the case of an ordinary sphere, S^2, Ricci $R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = 1/a^2$ (1/radius squared); so trace of Ricci = $2/a^2$ -- similar to ordinary Gaussian curvature}.* Both of these terms might be present in a candidate for an Einstein tensor, $G_{\mu\nu} = c_1R_{\mu\nu} + c_2\, g_{\mu\nu}R$ – And the Bianchi identities pin down the value of c_1 and c_2 ($c_2 = -c_1/2$, *Weinberg, p 153*). These in turn ensure that the divergence of $G_{\mu\nu}$ equals zero which then ensures that the divergence of $T_{\mu\nu} = 0$ – and that means that energyMomentum is conserved *(a supposed local requirement).* In other words, the Bianchi identities provide local *(not necessarily global)* conservation: covariant Div T = $\nabla \cdot$ T = $T^{\mu\nu}{}_{;\nu} = 0$. And this justifies including a scalar R term. After establishing the correct form for $G_{\mu\nu}$ (shown above), then the Bianchi identities can be stated "simply" as $G^{\nu}{}_{\mu\,;\nu} = 0$.

Einstein had earlier proposed the simpler gravitation field equation $R_{\mu\nu} = -\kappa T_{\mu\nu}$ *(eqn 14.13 p 253 Pais).* By 11/18/1915, he was still missing the trace term R; but by 11/25/15 he incorporated trace T, a form that is equivalent to using trace R (shown above). But he and Hilbert didn't yet know the Bianchi identities *until after 1917;* so, a bit of trial-and-error was involved. The symmetric tensor $G^{\mu\nu}$ has 10 independent components. Bianchi identities eliminate 4 leaving 6 independent non-linear equations. For many free space problems, $R_{\mu\nu} = 0$ is still adequate without the trace.

There is a broader "Cartan geometry" (1922) which is a generalization of Riemannian geometry that includes both curvature and torsion, and *its* Bianchi identity relates the curvature 2-form to the torsion form. A later inclusion of spinors (spin) naturally led to torsion, and Einstein's later "unified theories" were based on a non-symmetric metric tensor that allowed torsion. "Cartan theory is nonmetric but agrees with experiment and is experimentally indistinguishable from general relativity *(ref. MTW p 1068).* Einstein's theory of gravitation is quite adequate, but it is not the only workable theory.

Bianchi identities were of fundamental importance to finding the Einstein equation. But they were not very intuitive ! : take the covariant derivative (";") of the Riemann tensor $R_{\lambda\mu\nu\kappa:\eta}$ and then permute the ν, κ, and η to get 3 terms = to zero (Bianchi, 1902). For the special case of electromagnetism using the electromagnetic tensor $F_{\beta\gamma}$, $\partial_{[\alpha}F_{\beta\gamma]} = 0$ is a Bianchi identity. The modern mathematical language of "differential forms" makes it easier: **F = dA;** and $\mathbf{dF = d^2A = 0}$ is a Bianchi identity where A is the 4-potential 1-form [ϕ, **A**] = $\phi dt+A_xdx+A_ydy+A_zdz$. For curved-space relativity, find a curvature 2-form R^{ab} and take its "exterior Lorentz covariant derivative" $DR^{ab} = 0$ (a Bianchi identity; it looks simple in these symbols, but it takes a lot of study).

In place of *(the at that time unfamiliar)* Bianchi identities, Einstein instead used local conservation of energy as a constraint in forming his equations. His final proper field equations imply local energy/momentum conservation as $T^{\mu\nu}{}_{;\nu} = 0$. There is a nonlinear tradeoff between general covariance and conservation. "Non-gravitational energy/momentum creates gravitational energy/momentum." An extreme example is the rotating Kerr black hole where the rotating "dragged" vacuum possesses huge energy and angular momentum while matter may no longer be present. Both gravitational and non-gravitational energy/momentum have to be considered together (a non-covariant pseudo-tensor). The definition and evaluation of global energy momentum over extended regions has been a huge problem.

The *"Light-Quantum Hypothesis"* – EnglishTranslation of Einstein's photoelectric effect paper of 1905 {first appeared in Am J Physics in 1965:

http://astro1.panet.utoledo.edu/~ljc/PE_eng.pdf

Chapter 19, Abraham Pais book p 364.

What we call the Photo-Electric effect is only mentioned in the last two pages of Einstein's published paper.

Since Planck's Black Body equation is usually stated as the starting point of quantum mechanics, it is often assumed that Einstein's 1905 paper justified the "photon" quanta explicitly using that equation. But he instead discussed earlier **approximations** to Black Body radiation (BB, 1900) such as Wien's radiation formula.

$$\rho(\nu,T)_{Planck} = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}, \rightarrow \rho_{Wien} = \frac{8\pi h\nu^3}{c^3}e^{-h\nu/kT} \quad \& \quad \rightarrow \rho_{RJ} = \frac{8\pi\nu^2 kT}{c^3}$$

{Planck's BB is spectral density eqn. 19.6 on p. 368 and also eqn. 2.1 in the Pais book. RJ is eqn. 19.16 & 19.17, and Wein is eqn. 19.5 and19.7}. There is a difference factor of 4π/c between these CGS units to more modern SI ~ MKSA units.

Wien's Law of 1896 only applies when 'x' = $h\nu/kT \gg 1$ and contains the new constant "h" eventually noticed for the first time. Planck recognized that the constants h, c, and G could be combined to form special "Planck Units" in a publication in 1899 -- a year before his BB 1900 paper! Wien's approximation is generally not too bad *{e.g., see https://en.wikipedia.org/wiki/Rayleigh%E2%80%93Jeans_law}* but notably fails for radiation in the far infra-red (as shown in the Graph on page 367 of Pais for a fixed IR wavelength). In this plot, the Rayleigh-Jeans approximation works very well. In the high frequency limit of Planck's equation, x is large and $1/(e^x-1) \simeq 1/e^x = e^{-x}$ corresponding to the earlier "**Wien's** approximation" $\rho = \alpha\nu^3 e^{-h\nu/kT}$ (eqn. 19.5, although k= k_B = R/N as "Boltzmann's constant" and the symbol h as "Planck's constant" hadn't yet been used directly). This is the part used by Einstein for his "hypothesis." Solving this for temperature gives $1/T = -(k/h\nu)\ell n(\rho/\alpha\,\nu^3)$ *{used on page 176}.*

For the special case of low frequency, or high-wavelengths or temperatures, x = $h\nu/kT$ is small. Then the term $(e^x - 1) \simeq (1 + x + \ldots) - 1 \simeq x$, which puts a factor $kT/h\nu$ into the numerator of the Planck's equation eq.2.1 and lowers the power on the frequency to a ν^2. This is the old incorrect Rayleigh-Jeans classical spectral law approximation whose extension to high frequency leads to an "ultraviolet catastrophe."

The "Rayleigh-Jeans" ("RJ") 1900 & 1905 classical formula only applies to low frequencies or large wavelengths (far IR), and Einstein derived it correctly for the first time in 1905 *{ArXiv 0510180}.* Rayleigh only noted the $\propto \nu^2$ dependence in 1900, and experimentalists found $\rho \propto T$ also that year (a difficult discovery that motivated Planck). Einstein began with Planck's 1897 formula $\rho(\nu,T) = (8\Sigma\,\nu^2/c^3)U(\nu,T)$ for an electromagnetic oscillator *(eqn. 19.11 p 369).* U is an equilibrium energy = expectation value $\langle E \rangle$ *{eqn 20.2 p 395}.* For RJ, U = kT .

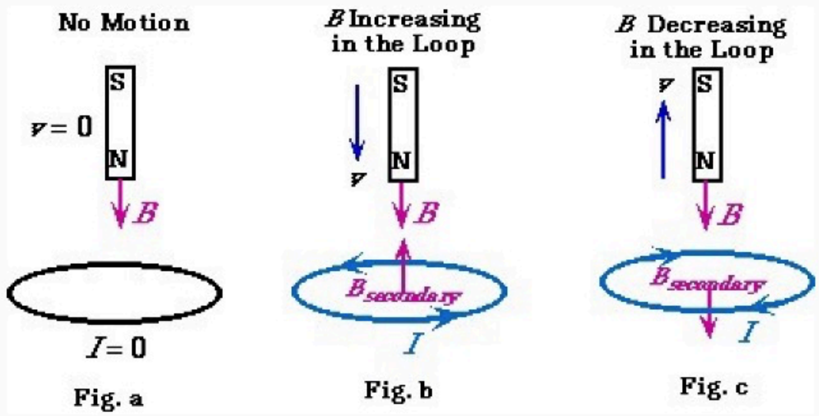
Einstein decided that the RJ formula failed because matter and radiation were not treated symmetrically: matter is discrete and radiation should also be discrete. He shows that Wien's non-classical formula valid at large ν/T behaves like a gas of radiation quanta. The concept of photons as "particles" became more explicit when their "momentum" was stated in 1917. And then the Compton effect of 1923 convinced most physicists.

Magnet and a Conducting Wire Loop Approaching Each Other:

One motivator for Einstein's famous Relativity paper of 1905, "Electrodynamics of Moving Bodies," was the "asymmetry of a system consisting of a magnet and a

conductor"
http://hermes.ffn.ub.es/luisnavarro/nuevo_maletin/Einstein_1905_relativity.pdf
{ The analysis of this elementary setup is very rich, is well worth dwelling on and was skimpy in our book, e.g., Pais p 140}.

Before relativity, E and B fields were considered as separately distinct real entities: there are cases with pure E fields and cases with only pure B fields. Einstein's paper stressed the conundrum of a magnet and a conducting wire loop in relative motion with an emf being induced in the conductor whether either the magnet or the loop are separately viewed as being in motion.



The concept of electromagnetic induction, illustrated via a bar magnet and a loop of wire. RICHARD VAWTER OF WESTERN WASHINGTON UNIVERSITY

For the case of magnet in motion, both an E field and a B field surround the magnet with the E field being due to B-field strength changing in space-time (Faraday's Law of Induction). But for the case of the conductor in motion, there is no E field about the magnet, and current in the wire is only due the "Lorentz" force **E = v×B** on electrons **in** the wire. The different views and the presence or absence of a surrounding E field was "**unbearable**" to Einstein! {but apparently to no one before him}. He claimed that "magneto-electric **induction** compelled him to postulate the principle of relativity." Induction was then "merely an artifact of motion relative to the observer" [pitt.edu]. "Maxwell's electrodynamics conformed to the principle of relativity" as long as Lorentz transformations are also included.

Faraday's Law of induction is often stated as Emf = - $d\Phi_B/dt$: a change in magnetic flux, Φ_B, through a conducting wire loop induces a back electromotive voltage and current whose resulting magnetic field opposes the change {see Figure above. The minus sign is called "Lenz's" law and should make one ponder "why"}. Physicists often prefer a differential form for Faraday's equation: $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$.

Magnetic Vector Potential View, A(x,t):

The utility of potentials has been in and out of favor over time. In early papers, Maxwell considered them basic and called the term "qA" "electromagnetic momentum" where q = electrical charge {this is similar to the inertial "kinetic momentum" p = mv}. Then there was a long period where only electromagnetic fields were basic. And then the "Aharonov-Bohm" AB effect of 1959 again made potentials seem more basic (electron phases change due to the presence of A fields, $\Delta\varphi = \int qA\cdot dx/\hbar$).

Faraday's Law becomes more transparent if we think in terms of potentials for E and B fields. A more primitive induction here is $\mathbf{E} = -\partial \mathbf{A}/\partial t$ *{and also $-\nabla\phi$ when electrostatic potentials are present}* where **A** is the magnetic vector potential such that the magnetic field is $B = \nabla\times A$.

A "**cute**" way to "derive" Faraday's equation is to begin with the **A** potential. Create an operator $\mathbf{D} = \nabla\times(\partial/\partial t)$ of something – which also = $(\partial/\partial t)\nabla\times$___ {the ordering of space versus time differentiation makes no difference}. Faraday's law is the consequence of the equality of both orderings – space and time).

That is: $\mathbf{DA} = \mathbf{D}\,\mathbf{A} = (\partial/\partial t)\nabla\times A = \partial B/\partial t = \nabla\times(\partial A/\partial t) = \nabla\times(-E)$.

So, $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$

– Faraday's Law of Induction *{Faraday discovered induced current in 1831}. {This derivation may be helpful to those of us who occasionally forget Maxwell's equations – which we all do from time to time}.*

It is textbook convention to say that "all magnetic fields encountered in nature are generated by circulating surface currents" – like wire-wound solenoid-like currents {e.g., Pais, p 247}. An ensemble of little rotating currents would be canceled out inside a magnet but cannot cancel at its surface. The field source (for H or B) could then be replaced by current through a **solenoid** in place of the magnet surface {Ampere, 1827}. **But**, electron spin is a dominant source of magnetic field for permanent magnets (e.g., Pais, p 248); and a rotating electron cannot readily be considered as a rotating electric current. But, spinning electrons can be considered as having a rotating A-field (at least at a suitable distance from classical electron size). So "surface currents" perhaps should be replaced by rotating surface vector **A** fields instead. This is relatively unaffected by whether a magnet is moving or not and cannot be transformed away. The magnet frame knows it has an A and a B field, and a moving conductor also sees these fields.

In the frame of a conductor "at rest" and a magnet approaching it with a speed v *{co-axially as in the Figure above},* the wire loop experiences the field from the magnet to be changing with the separation distances from a pole-face. So the conductor experiences **induction** on its conduction electrons due to $F = qE = -|e|\, dA/dt$ {or equivalently a changing magnetic field B with changing flux through a wire loop}. This E field is also deduced to exist all about the magnet, and it actually serves to drive a current flow in the presence of the wire loop.

In the frame of the magnet at rest and wire loop approaching, it a conduction electron has the velocity v and experiences a Lorentz force $F = q\mathbf{v}\times\mathbf{B}$ (-- a velocity in one direction crossed with a B field in another direction results in a force in a third direction -- which may *{and initially should}* seem "strange"). What other physical effect looks like this? – a Coriolis force: $F = -2m(\boldsymbol{\omega}\times\mathbf{v})$. Here the observer is inside a rotating frame of reference with angular speed omega {…say winds rising north from the equator will be seen to curve by a weatherman on Earth}. A simpler example is a flat rotating LP record in Cylindrical coordinates with rotating speed $v_\phi = \rho\omega_o$. Then, $\nabla\times\mathbf{v} = (1/\rho)[(\partial/\partial\rho)(\rho v_\phi = \rho^2\omega_o)] = 2\omega_o$ – a curl is often like a local rotation. A bug trying to move radially will be seen to curve inside the rotating frame.

{Ignoring gauge freedoms} an A-field may be thought of as sort of a dragging of electromagnetic space due to moving currents (similar to a 1^{st}- order "Lense-Thirring" dragging of inertial space by moving masses in general relativity). A useful dragging formula was developed in part by Alfred-Marie Liénard in 1898 and independently by Emil Wiechert in 1900 approximately as:

$\mathbf{A} = (\mu_o/4\pi) \int \mathbf{J}(r',t') /|r-r'|d^3r' + A_o(r,t)$

{where J is vector electric current density, and vector A is dragged by J}. For just a single charge, $\mathbf{A} = \mu_o e\mathbf{v}/4\pi r$ *{falling off as 1/r. But for an observer traveling with the charge, there is no A field – it is a "relative" effect. But quantum-mechanics's wavenumber k = mv/ħ is also a relative effect}.* $B = \nabla \times A$, and A is $\propto$ v; so B at some location can be pictured as a local rotation, ω. An electron possesses **both** mass m and charge e and hence responds to both inertia and to eA . Unlike the Coriolis effect viewed from inside a rotating frame, we see the electron move in a magnetic field as if the electron partly lived in a rotating electromagnetic space; and we now view the electron in our Euclidean Lab space that doesn't 'see' the invisible A-field.

The "**A**-view" works, but $F = qv\times B$ is a bit easier for calculations. We still have two different discussions with A: induction in one frame and Coriolis-like contributions in another frame. The mathematics of the relativistic electromagnetic tensor $F_{\mu\nu}$ unifies the different views (but that had to await Minkowski in 1908).

{Ref: https://www.pitt.edu/~jdnorton/Goodies/magnet_and_conductor/index.html}.

https://en.wikipedia.org/wiki/Moving_magnet_and_conductor_problem

Why the minus sign Lenz's Law?

Faraday's law includes Lenz's law that the sign of the induced electric field is negative: $\nabla \times E = -\partial B/\partial t$. If the magnetic field near a conductor is seen as increasing, then the induced current in the conductor will produce its own magnetic field in the opposing direction as if magnetic fields had some sort of inertia. If the vector potential A is viewed as more fundamental, then the E field induction is $= -\partial A/\partial t$. Why the minus sign?

In many textbooks, Lenz's Law is simply assumed and applied to problems. It is sometimes also stated that conservation of energy requires it because a plus sign would enhance a magnetic field and produce runaway energy – a somewhat "hand-waving" argument. Another justification says that one must have a relativistic view and re-write Maxwell's equations using the field strength tensor $F_{\mu\nu}$ *(which was found by Minkowski in 1908).*

But another approach is using a classical velocity dependent potential $U = q(\phi - v\cdot A)$ appropriate for discussing the velocity dependent Lorentz force $F = q(-\nabla\phi + v\times B)$.

[Combined "Canonical Momentum" $\Pi = p + qA] = -q\nabla(\phi - v\cdot A)$, with $p = \gamma mv$. But, in our problem, electrostatic potential $\phi = 0$ and $v \perp A$ because A is a rotational field, $A = A_\theta$. So $v\cdot A = 0$ and "canonical momentum" Π is conserved. Then $(\partial/\partial t)\Pi = 0$.

{For convenience, ignore Ben Franklin's error about what is negative and positive and consider 'positive' charges}.

For a movable charge in a conducting wire, $p = mv_\theta = -qA$, and Force $F_\theta = dp/dt = ma_\theta = -q\,\partial A/\partial t = qE_\theta$ (the induction equation with the minus sign). Charges accelerating in an angular direction result in current flow in the conducting wire loop.