CHAPTER 8

8.1 $$\left.\frac{U^{235}}{U^{238}}\right|_{final} = \left.\frac{U^{235}}{U^{238}}\right|_{initial} \cdot \exp(\lambda(U^{238}) - \lambda(U^{235}))t$$

$$\ln\left(\frac{7.2\times10^{-3}}{1.65}\right) = (0.15 - 0.97)\times10^{-9}\ t$$

$$t_{GALAXY} = (-5.34)/(-0.82\times10^{-9}) \boxed{= 6.6\times10^{9}\ yr}$$

ADD 1 BILLION YRS,

$$t_{UNIV} = 7.6\times10^{9}\ yr$$

EQN. (8.5) GIVES

$$t_0 = 6.51\ h^{-1}\times10^{9}\ yr$$
$$= 7.6\times10^{9}\ yr$$

SO,

$$\boxed{h \leq 0.86}$$

8.2 EMPTY UNIVERSE: $\Omega_0 \to 0$

$$\lim_{\Omega_0\to0} \cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right) \cong \lim_{\Omega_0\to0} \ln\left(\frac{4-2\Omega_0}{\Omega_0}\right)$$

$$= \lim_{\Omega_0\to0}\left[\ln(4-2\Omega_0) - \ln\Omega_0\right]$$

$$= \ln4 - \lim_{\Omega_0\to0} \ln\Omega_0$$

$$= \ln4 - \lim_{\Omega_0\to0}\left(\sum_{n=1}^{\infty}(-1)^n\frac{(\Omega_0-1)^n}{n}\right)$$

$$= \ln4 - \underbrace{\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)^n}_{\text{THIS SERIES IS THE ALTERNATING HARMONIC SERIES} = \ln2}$$

$$= \ln4 - \ln2 = \ln2$$

THIS IS FINITE, SO

8.2 Cont

$$\boxed{\lim_{\Omega_0 \to 0} H_0 t_0 = 1 - (0)(\ln 2) = \underline{1}}$$

FLAT UNIVERSE: $\Omega_0 \to \underline{1}$

IF $x = 1 - \Omega_0$, THEN AS $\Omega_0 \to 1$, $x \to 0$.

SO WRITE,

$$H_0 t_0 = \frac{1}{x} - \frac{1-x}{2x^{3/2}} \cosh^{-1}\left(\frac{1+x}{1-x}\right) \text{, FOR SMALL } x,$$

$$\cong \frac{1}{x} - \frac{1-x}{2x^{3/2}} \left[2\sqrt{x} + \frac{2x^{3/2}}{3}\right]$$

$$\cong \frac{1}{x} - \frac{2x^{1/2} - 2x^{3/2} + \frac{2}{3}x^{3/2} - \frac{2}{3}x^{5/2}}{2x^{3/2}}$$

$$\cong \frac{1}{x} - \frac{1}{x} + \frac{2}{3} + \frac{1}{3}x = \frac{2}{3} + \frac{x}{3}$$

$$\boxed{\lim_{x \to 0} H_0 t_0 = 2/3}$$

8.3 SEE PROBLEM ANSWERS IN BOOK

8.4 FORM THE RATIO: $\frac{H^2}{H_0^2} = \frac{\frac{8\pi G}{3}(\rho + \rho_\Lambda)}{\frac{8\pi G}{3}(\rho_0 + \rho_{\Lambda_0})} = \frac{\left(\frac{\dot{a}}{a}\right)^2}{H_0^2}$ MULTIPLY BY $\frac{1/\rho_c(t_0)}{1/\rho_c(t_0)}$ &

SOLVE FOR $\dot{a}^2$;

$$\dot{a}^2 = H_0^2 a^2 \left[\frac{\rho/\rho_c(t_0) + \rho_\Lambda/\rho_c(t_0)}{\rho_0/\rho_c(t_0) + \rho_\Lambda(t_0)/\rho_c(t_0)}\right]$$

SINCE $\Omega_0 = \rho_0/\rho_c(t_0)$; $\Omega_\Lambda(t_0) = 1 - \Omega_0 = \rho_\Lambda(t_0)/\rho_c(t_0)$ AND

$\rho_\Lambda = \rho_\Lambda(t_0)$; $\rho = \rho_0/a^3 = \Omega_0 \rho_c(t_0)/a^3$

$$\dot{a}^2 = H_0^2 a^2 \left[\frac{\Omega_0/a^3 + (1-\Omega_0)}{\Omega_0 + (1-\Omega_0)}\right] = \boxed{H_0^2\left[\Omega_0 a^{-1} + (1-\Omega_0)a^2\right]}$$

8.4 cont.

$$\dot{a} = \frac{da}{dt} = H_0 \sqrt{\frac{\Omega_0}{a} + (1-\Omega_0)a^2}$$

SOLVE FOR dt AND SUBSTITUTE INTO $t_0 = \int_0^{t_0} dt$:

$$H_0 t_0 = H_0 \int_0^{t_0} dt = \int_0^1 \frac{a^{1/2}\,da}{\sqrt{\Omega_0 + (1-\Omega_0)a^3}}$$

[NOTE: LIMITS ARE FOUND USING (8.3). AS t GOES FROM 0 TO t_0, a GOES FROM 0 TO 1.]

$$H_0 t_0 = \frac{1}{\sqrt{1-\Omega_0}} \int_0^1 \frac{a^{1/2}\,da}{\sqrt{b + a^3}} \quad \text{WITH } b = \Omega_0/(1-\Omega_0)$$

TO DO THE INTEGRAL, LET $du = a^{1/2}da$, SO $u = \frac{2}{3}a^{3/2}$ (THE INTEGRATION CONSTANT WILL SUBTRACT OUT IN THE DEFINITE INTEGRAL.) AND $a^3 = \frac{9}{4}u^2$. SO,

$$H_0 t_0 = \frac{1}{\sqrt{1-\Omega_0}} \int_0^{2/3} \frac{du}{\sqrt{b + \frac{9}{4}u^2}} = \frac{2}{3\sqrt{1-\Omega_0}} \int_0^{2/3} \frac{du}{\sqrt{b' + u^2}}$$

WHERE $b' = \frac{4}{9}b$. FROM TABLES

$$\int \frac{dx}{\sqrt{c^2 + x^2}} = \sinh^{-1}\left(\frac{x}{|c|}\right), \text{ SO}$$

$$H_0 t_0 = \frac{2}{3\sqrt{1-\Omega_0}} \left[\sinh^{-1} \frac{u}{\frac{2}{3}\sqrt{\frac{\Omega_0}{(1-\Omega_0)}}} \right|_0^{2/3}$$

$$= \boxed{\frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}} \sinh^{-1}\left[\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right]}$$

BUT, HOW IS $\sinh^{-1}$ RELATED TO $\ln$?

8.4 Cont.

JUST AS THE HYPERBOLIC TRIG FUNCTIONS CAN BE WRITTEN IN TERMS OF EXPONENTIALS, THEIR INVERSES CAN BE WRITTEN AS NATURAL LOGS.

IF $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$, THEN MULTIPLYING BY e^x

$$e^{2x} - 2ye^x - 1 = 0.$$

THE ROOTS OF THIS QUADRATIC EQUATION ARE,

$$e^x = y + \sqrt{y^2+1}$$

SINCE $x = \sinh^{-1} y$, TAKING ln OF BOTH SIDES

$$\sinh^{-1} y = \ln\left(y + \sqrt{y^2+1}\right).$$

FOR US, $y = \sqrt{\dfrac{1-\Omega_0}{\Omega_0}}$, SO

$$\sinh^{-1} y = \sinh^{-1}\left[\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right]$$

$$= \ln\left[\sqrt{\frac{1-\Omega_0}{\Omega_0}} + \sqrt{\frac{1}{\Omega_0}}\right]$$

$$= \ln\left[\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right]$$

AND WE CAN WRITE THE OTHER EQUALITY IN (8.6).

CHAPTER 9

9.1 $\Omega_{rad} = \rho_{rad}/\rho_c$

SINCE $\#\gamma = \#\nu$, THEIR NUMBER DENSITIES ARE ALSO EQUAL, $n_\gamma = n_\nu$

$$\rho_{rad} = n_\gamma E_\gamma \text{ AND } \rho_\nu = n_\nu E_\nu = n_\gamma E_\nu$$

IF NEUTRINOS MAKE UP THE CRITICAL MASS, $\rho_\nu = \rho_c$, SO

$$\Omega_{rad} = n_\gamma E_\gamma / n_\gamma E_\nu = E_\gamma / E_\nu \Rightarrow E_\nu = E_\gamma / \Omega_{rad}$$

SINCE E_γ IS ALL THERMAL, $E_\gamma = 3 k_B T$
AND E_ν IS ALL MASS ENERGY,

$$E_\nu = (3)(8.6 \times 10^{-5}\,\text{eV K}^{-1})(2.725\,\text{K}) / 2.47 \times 10^{-5} h^{-2}$$
$$= \boxed{28 h^2\,\text{eV}}$$

IF $E_\nu = 10\,\text{eV}$, $h \le \sqrt{\frac{10}{28}} = 0.60$
IF $E_\nu = 90\,\text{eV}$, $h \le \sqrt{\frac{10}{90}} = 0.33$

EQN (6.2) GIVES $h = 0.72 \pm 0.08$, SO NEITHER E_ν VALUE IS WITHIN ERROR BOUNDS. LATER OBSERVATIONS HAVE NARROWED ERROR BOUNDS STILL FURTHER TO $h = 0.72 \pm 0.02$. THUS, WHILE NEUTRINOS MAY CONTRIBUTE TO DARK MATTER, THEY ARE NOT THE WHOLE STORY.

9.2 AN INDIVIDUAL BLACK HOLE HAS A MASS OF $M_{BH} = 10^{-10} M_\odot$ AND THEY ARE EVENLY DISTRIBUTED IN A HALO OF DIAMETER 10^5 LY $= 3.1 \times 10^{-2}$ Mpc, OR VOLUME OF $V_{HALO} = \frac{4}{3}\pi(1.5\times10^{-2})^3\ Mpc^3 = 1.4\times10^{13}\ pc^3$.

IF THESE BLACK HOLES ARE TO ACCOUNT FOR THE CRITICAL MASS DENSITY WITH 1 GALAXY PER Mpc^3, THEY MUST HAVE A TOTAL MASS OF $M_c = (\rho_c)(1\ Mpc^3) = (2.78\ h^2 \times 10^{11}\ M_\odot/Mpc^3)(1\ Mpc^3)$

$= 1.44\times10^{11}\ M_\odot$ (TAKING $h = 0.72$)

THUS THE DENSITY OF HALO BLACK HOLES MUST BE SUCH THAT ON AVERAGE ONE BLACK HOLE APPEARS IN EACH VOLUME OF:

$$V_{BH} = \frac{M_{BH}}{M_c} \cdot V_{HALO}$$

$$= \frac{10^{-10}\ M_\odot}{1.44\times10^{11}\ M_\odot} \cdot 1.4\times10^{13}\ pc^3$$

$$= 9.8\times10^{-9}\ pc^3$$

OR AN AVERAGE SEPARATION OF:

$$d = \sqrt[3]{9.8\times10^{-9}}\ pc \boxed{\begin{aligned} &= 2.1\times10^{-3}\ pc \\ &= 6.5\times10^{10}\ km \end{aligned}}$$

OR ABOUT 10X THE DIAMETER OF PLUTO'S ORBIT.

CHAPTER 10

10.1 THE TEMP OF THE μWAVE RADIATION SIMPLY REFLECTS THE MODAL ENERGY, $E = h\nu$, OF THE RADIATION.

A μWAVE OVEN HEATS FOOD BY THE PROCESS OF DIELECTRIC HEATING IN WHICH THE POLAR WATER MOLECULES (THAT EXHIBIT A DIPOLE MOMENT) RE-ORIENT THEMSELVES AT GHz RATES IN RESPONSE TO THE OSCILLATORY E-FIELD OF THE μWAVE RADIATION. THIS MOLECULAR VIBRATIONAL ENERGY APPEARS AS HEAT.

10.2 $a(t) \propto t^{1/2} \Rightarrow \dot{a}(t) \propto 1/2\sqrt{t}$, SO THE FRIEDMANN EQUATION (ASSUMING A FLAT GEOMETRY AND $\Lambda = 0$) IS:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_{rad}}{3} = \frac{1}{4t^2} \Rightarrow \rho_{rad} = \frac{3}{32\pi G t^2}$$

SINCE $\rho_{rad} c^2 = \alpha T^4$, $T^4 = 3c^2/32\pi G t^2 \alpha$

$$\text{OR } T = \left[\frac{(3)(3\times10^8\ m\,s^{-1})^2(1\ J\,s^2\,Kg^{-1}\,m^{-2})}{(32)(\pi)(6.67\times10^{-11}\ m^3\,Kg^{-1}\,s^{-2})(1\,s)^2(7.57\times10^{-16}\,J\,m^{-3}\,K^{-4})}\right]^{1/4}$$

$$= \left(5.3\times10^{40}\,K^4\right)^{1/4} = \boxed{1.5\times10^{10}\ K}$$

$$\rho(t=1sec) = \rho_0\left(\frac{t_0}{t}\right)^2 = \frac{\Omega_0 \rho_c t_0^2}{1sec^2} = (0.3)(1.88\,h^2\times10^{-26}\,Kg\,m^{-3})(4.3\times10^{17})^2$$

$\rho(1sec) = 5.5\times10^{8}\ Kg\,m^{-3}$.

NOT SURE WHERE THE DISCREPANCY WITH LIDDLE'S ANSWER OF $2\times10^9\ Kg\,m^{-3}$ ARISES.

10.2 cont. $\rho_{H_2O} = 10^3\ Kg\ m^{-3}$, so $\rho(t=1sec) = \boxed{5.5\times10^5 \text{ TIMES } \rho_{H_2O}}$

$\rho(t) = 10^3\ Kg\,m^{-3} = \dfrac{\Omega_0 \rho_c t_0^2}{t^2} \Rightarrow t = \left(\dfrac{\Omega_0 \rho_c t_0^2}{10^3}\right)^{1/2} = \left(\dfrac{5.5\times10^8}{10^3}\right)^{1/2}$

$\boxed{t = 740 \text{ sec}}$

10.3 SINCE THE EQUATIONS OF UNIVERSAL EVOLUTION ARE TIME REVERSIBLE, WHEN $a(t) = a_0$ DURING THE "CRUNCH" PHASE, $T = T_0 = 2.725\,K$.

10.4 $n_{0,e^-} = 0.2 m^{-2}$; $a(t') = a_0 \times 10^{-6}$

$n_{e^-}(t') = n_0 \left(\frac{a_0}{a(t')}\right)^3 = (0.2 m^{-3})(10^6)^3 = \boxed{2\times10^{17}\ m^{-3}}$

For e^-, $mc^2 = 0.511$ Mev $= 5\times10^5$ eV

SINCE (10.7), $T \propto a^{-1}$, $T(t') = T_0 \times 10^6 = 2.7\times10^6\,K$

AND $k_B T = (8.6\times10^{-5} eV K^{-1})(2.7\times10^6) K = 230$ eV

SINCE $230 eV << 5\times10^5 eV$, $k_B T << mc^2 \rightarrow$ $\boxed{\text{NONRELATIVISTIC}}$

$d \cong (n(t')\cdot\sigma_e)^{-1} = [2\times10^{17} m^{-3} \times 6.7\times10^{-29} m^2]^{-1} \cong \boxed{7.7\times10^{10} m}$

$\Delta t = d/c = 7.7\times10^{10} m / 3\times10^8\ ms^{-1} = \boxed{260 \text{ sec} = 4.2 \text{ min}}$

$\Delta t << t_{univ} \cong 10{,}000$ yr $\rightarrow$ NO γ FROM THAT ERA EXIST TODAY.

10.5 VERIFY THE CONDITION $I >> k_B T$ IS VALID:

$k_B T \cong (8.6\times10^{-5} eV\,K^{-1})(\sim10^4\,K) \cong 1 eV$

SINCE IONIZATION ENERGY FOR H IS 13.6 eV

AN ORDER OF MAGNITUDE SHOULD BE SUFFICIENT.

$n_\gamma = 3.7\times10^8\ m^{-3}$ (10.11)

$n_B = 0.22\ m^{-3}$ (10.14)

IF $n_\gamma(>I) = n_B$,

$\dfrac{n_\gamma(>I)}{n_\gamma} = \dfrac{n_B}{n_\gamma} = \dfrac{0.22 m^{-3}}{3.7\times10^8 m^{-3}} = 5.9\times10^{-10}$

10.5 Cont.

FORM $\frac{I}{k_B} = \frac{13.6\,eV}{8.6\times10^{-5}\,eV\,K^{-1}} = 1.6\times10^5\,K$

SO, WE REQUIRE

$$5.9\times10^{-10} = \left(\frac{1.6\times10^5\,K}{T\,K}\right)^2 \exp\left(-\frac{1.6\times10^5\,K}{T\,K}\right), \text{ OR}$$

$$2.4\times10^{-20}\,T^2 = \exp\left(-\frac{1.6\times10^5}{T}\right)$$

TAKING ln OF BOTH SIDES

$$\ln(2.4\times10^{-20}) + 2\ln T = \left(-\frac{1.6\times10^5}{T}\right)$$

SOLVING NUMERICALLY WITH MATHEMATICA:

```
NSolve[Log[2.4*^-20] + 2 * Log[T] == -1.6*^5 / T, T]
```

NSolve::ifun : Inverse functions are being used by NSolve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[17]= $\{\boxed{\{T \to 5741.94\}}, \{T \to 6.45489\times10^9\}\}$

OR $T \approx 5700\,K$.

10.6 IF WE USE THE AGE CALCULATED IN (8.5)

$t_0 = 6.51 h^{-1}\times10^9$ YRS

AND LIGHT SPEED OF 3.08×10^{-7} Mpc yr^{-1}

THE RADIUS OF THE LAST SCATTERING SURFACE IS

$$\boxed{r_{LS} = (6.51\,h^{-1}\times10^9\,yr)(3.08\times10^{-7}\,Mpc\,yr^{-1}) \cong 2000\,Mpc}$$

IN MY NOTES TO LIDDLE, I POINT OUT THAT WE REALLY SHOULD USE THE CONFORMAL TIME, η_0,

10.6 Cont.

SINCE IT ACCOUNTS FOR THE EXPANSION THAT HAS OCCURRED SINCE THE PHOTONS γ BEGAN THEIR TRAVELS. DOING SO WOULD INCREASE r_{LS} BY A FACTOR OF ABOUT 3.

THE CRITICAL DENSITY $\rho_c(t_0) = 2.78\, h^2 \times 10^{11}\, M_\odot\, Mpc^{-3}$ (6.6) SPREAD OVER A VOLUME $V_{UNIV} = \frac{4}{3}\pi r_{LS}^3$ GIVES A TOTAL MASS OF THE UNIVERSE OF

$$M_{UNIV} = \rho_c(t_0) V_{UNIV} = \frac{4\pi}{3}(6000\, h^{-1}\, Mpc)^3 (2.78\, h^2 \times 10^{11}\, M_\odot Mpc^{-3})$$
$$= 2.5\, h^{-1} \times 10^{23}\ M_\odot$$

IF $\Omega_{0,BAR} = 0.044$, THE BARYONIC MASS IS

$$M_{UNIV,BAR} = 0.044\, M_{UNIV} = 1.1\, h^{-1} \times 10^{22}\ M_\odot$$

IF EACH GALAXY CONTAINS $10^{11}\ M_\odot$, THEN THERE ARE

$$\#GAL = \frac{1.1 \times 10^{22} M_\odot}{10^{11}\ M_\odot} \cong \boxed{10^{11}\ \text{GALAXIES}}$$

ASSUMING # PROTONS $\cong$ # NEUTRONS AND $M_p \cong M_n$,

$$\#\,\text{PROTONS} = \frac{1}{2}\frac{(\#\odot)(M_\odot)}{M_p}$$

$$= \frac{(0.5)(1.1 \times 10^{22})(2 \times 10^{30}\, Kg)(3 \times 10^{8}\, m\, s^{-1})^2}{(9.4 \times 10^{8}\, eV\, proton^{-1})(1.6 \times 10^{-19}\, J\, eV^{-1})(1\, Kg\, m\, s^{-2}\, J^{-1})}$$

$$= 7 \times 10^{(22+30+16+19-8-1)} = \boxed{7 \times 10^{78}\ \text{PROTONS}}$$