CHAPTER 6

6.1 IN A RADIATION-DOMINATED UNIVERSE, $P = \rho c^2/3$, SO THE ACCELERATION EQUATION BECOMES,

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\left(\rho + \frac{3}{c^2}\cdot\frac{\rho c^2}{3}\right) = \frac{-8\pi G}{3}\rho$$

AND THE CRITICAL DENSITY EQUATION (6.4) IS

$$\rho_{0,CR} = \frac{3H_0^2}{8\pi G} \Rightarrow H_0^2 = \frac{8\pi G}{3}\rho_{0,CR}$$

SO,

$$q_0 = -\frac{\ddot{a}_0}{a_0}\cdot\frac{1}{H_0^2} = -\left(-\frac{8\pi G}{3}\rho_0\right)\cdot\left(\frac{3}{8\pi G\rho_{0,CR}}\right) = \frac{\rho_0}{\rho_{0,CR}} = \boxed{\Omega_0}$$

6.2

$$q_0 = -\frac{a_0\ddot{a}_0}{\dot{a}_0^2} < 0 \Rightarrow \frac{a_0\ddot{a}_0}{\dot{a}_0^2} > 0$$

IN AN EXPANDING UNIVERSE $a_0 > 0$ AND $\dot{a}_0 > 0$. FOR THIS INEQUALITY TO BE SATISFIED, $\ddot{a}_0 > 0$. THE ACCELERATION EQUATION REQUIRES

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

SINCE LHS >0 AND $G > 0$, $\rho + \frac{3P}{c^2} < 0$. BUT $\rho > 0$, SO $\boxed{P < -\frac{\rho c^2}{3}}$. PRESSURE MUST BE SUFFICIENTLY NEGATIVE. SO, IF $q_0 < 0$, IMPLYING ACCELERATING EXPANSION AS HAS BEEN OBSERVED, THE OVERALL PRESSURE IN THE UNIVERSE MUST HAVE DROPPED BELOW THE $P=0$ OF A MATTER-DOMINATED CONDITION TO ONE DOMINATED BY THE COSMOLOGICAL CONSTANT.

CHAPTER 7

7.1 RADIATION ($\rho_{RAD} \propto 1/a^4$) AND MATTER ($\rho_{MAT} \propto 1/a^3$) DEPENDENCIES HAVE ALREADY BEEN FOUND (CH 5). FOR A COSMOLOGICAL CONSTANT, Λ, $\rho_\Lambda = \Lambda/8\pi G =$ A CONSTANT. BECAUSE OF THE $-K/a^2$ TERM IN THE FRIEDMANN EQUATION, NEGATIVE CURVATURE ($K<0$) HAS A $1/a^2$ DEPENDENCE. AT SUFFICIENTLY LARGE t:

$$\text{CONSTANT} > 1/a^2(t) > 1/a^3(t) > 1/a^4(t),$$

SO THE COSMOLOGICAL CONSTANT DOMINATES AT LATE TIMES. THE REVERSE IS TRUE FOR EARLY TIMES, WITH RADIATION BEING DOMINANT.

7.2 A PRESSURELESS UNIVERSE $\rightarrow P=0$

A STATIC UNIVERSE $\rightarrow \dot{a} = \ddot{a} = 0$

A CLOSED UNIVERSE $\rightarrow K > 0$

SO THE FRIEDMANN EQN. SAYS:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3} = 0$$

OR $\quad \Lambda = \frac{3K}{a^2} - 8\pi G\rho.$

AND THE ACCELERATION EQN. SAYS:

$$\frac{\ddot{a}}{a} = \frac{-4\pi G\rho}{3} + \frac{\Lambda}{3} = 0$$

OR $\Lambda = 4\pi G\rho.$

SO, $\quad 4\pi G\rho = \frac{3K}{a^2} - 8\pi G\rho \Rightarrow K = 4\pi G\rho a^2$

SINCE $K>0$ AND $a^2>0$, ρ MUST BE POSITIVE AND SO MUST $\boxed{\Lambda = 4\pi G\rho > 0}$

7.2 cont

SO EINSTEIN'S UNIVERSE HAS MASS-ENERGY DENSITY $\rho = \Lambda / 4\pi G$ AND FRIEDMANN'S EQUATION READS

$$\frac{8\pi G}{3}\left(\frac{\Lambda}{4\pi G}\right) - \frac{K}{a^2} + \frac{\Lambda}{3} = 0 \Rightarrow \text{ITS RADIUS IS } a = \sqrt{\frac{K}{\Lambda}}$$

SHOW THIS UNIVERSE IS UNSTABLE:

SUPPOSE A SLIGHT VARIATION IN Λ OCCURS: $\Lambda \to \Lambda + \Delta\Lambda$, SUBSTITUTING FOR ρ AND a AS ABOVE INTO FRIEDMANN EQN:

$$\frac{\dot{a}^2 (4\pi G\rho + \Delta\Lambda)}{K} = \frac{8\pi G\rho}{3} - \frac{K}{a^2} + \left(\frac{4\pi G\rho}{3} + \frac{\Delta\Lambda}{3}\right)$$

REARRANGING GIVES:

$$\dot{a}^2 = K\left(\frac{1}{1 + \frac{4\pi G\rho}{\Delta\Lambda}}\right)$$

NOW, $\dot{a}^2$ MUST BE ≥ 0 FOR $\dot{a}$ TO BE REAL, SO CONSIDER 3 CASES

1. $\dot{a} = 0 \to \Delta\Lambda = 0$. STATIC CASE. NO PORTURBATION
2. $\dot{a} < 0 \to -4\pi G\rho < \Delta\Lambda < 0$. THIS CASE COLLAPSES
3. $\dot{a} > 0 \to \Delta\Lambda > 0$ OR $\Delta\Lambda < -4\pi G\rho$. THIS IS THE EXPANDING CASE.

THUS THE EQUILIBRIUM OF EINSTEIN'S STATIC UNIVERSE IS UNSTABLE, AND A SLIGHT DEVIATION OF Λ FROM EXACTLY $4\pi G\rho$ WOULD CAUSE IT TO COLLAPSE OR EXPAND.

7.3 BY DEFINITION (6.14) $q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)} \cdot \frac{1}{H_0^2}$

THE ACCELERATION EQUATION REQUIRES:

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\rho(t) + \frac{\Lambda(t)}{3} \quad (\text{WITH } P=0)$$

FROM (7.3) $\frac{1}{H_0^2} = \frac{3\Omega_\Lambda}{\Lambda}$, SO

$$q_0 = \left(\frac{4\pi G \rho_0}{3} - \frac{\Lambda}{3}\right)\left(\frac{3\Omega_\Lambda}{\Lambda}\right)$$

BUT (7.6) GIVES $\Lambda = 8\pi G \rho_\Lambda$ AND $\Omega_\Lambda = \rho_\Lambda / \rho_c$, SO

$$q_0 = \left(\frac{4\pi G\rho_0 - 8\pi G \rho_\Lambda}{3}\right)\left(\frac{3\rho_\Lambda}{\rho_c(8\pi G \rho_\Lambda)}\right)$$

$$= \left(\frac{\cancel{4\pi G}(\rho_0 - 2\rho_\Lambda)}{\cancel{3}}\right)\left(\frac{\cancel{3}}{2\rho_c(\cancel{4\pi G})}\right) = \frac{1}{2}\frac{\rho_0}{\rho_c} - \frac{\rho_\Lambda}{\rho_c}$$

$$\boxed{q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda}$$

7.4 AN EXPANSION OF 5X IMPLIES THAT AT TIME t', $a(t') = 5a_0$. SINCE $\rho \propto 1/a^3$ (5.12), $\rho(t') = \rho(t_0)/125$. BUT, SINCE $\rho_\Lambda = \Lambda/8\pi G$, $\rho_\Lambda(t') = \rho_\Lambda(t_0)$.

THE CRITICAL DENSITY ALSO VARIES IN TIME AS $\rho_c(t) = \dfrac{3H^2(t)}{8\pi G}$. BUT $H^2(t) = \dfrac{8\pi G}{3}(\rho(t) + \rho_\Lambda(t))$, (7.7)

SO $\rho_c(t) = \rho(t) + \rho_\Lambda$ AND SO, $\Omega(t) = \dfrac{\rho(t)}{\rho_c(t)} = \dfrac{\rho(t)}{\rho(t)+\rho_\Lambda}$.

HENCE $\dfrac{\Omega(t')}{\Omega_0} = \dfrac{\rho(t')}{\rho(t')+\rho_\Lambda} \cdot \dfrac{\rho_0+\rho_\Lambda}{\rho_0}$

SINCE $\rho(t') \ll \rho_\Lambda$

$$\Omega(t') \approx \frac{(\rho_0/125)(\rho_0+\rho_\Lambda)}{\rho_0\rho_\Lambda}\Omega_0 = \left(\frac{\rho_0}{125\rho_\Lambda} + \frac{1}{125}\right)\Omega_0$$

SINCE CURRENTLY $\Omega_0 = \dfrac{\rho_0}{\rho_c(t_0)} = 0.3$ AND $\Omega_\Lambda = \dfrac{\rho_\Lambda}{\rho_c(t_0)} = 0.7$

$\dfrac{\rho_0}{\rho_\Lambda} = \dfrac{0.3}{0.7} = 0.4$, AND $\boxed{\Omega(t') = \left(\frac{0.4}{125} + \frac{1}{125}\right)(0.3) = 0.003}$

AND, SINCE $\Omega(t) + \Omega_\Lambda = 1$, $\boxed{\Omega_\Lambda = 0.997.}$

APPROXIMATELY, AT LATE TIMES, $\Omega \cong 0$ AND $\Omega_\Lambda \cong 1$. SINCE $\Omega \to 0$, $\rho \to 0$ AND, SINCE $K=0$, THE FRIEDMAN EQN. BECOMES $\left(\dfrac{\dot a}{a}\right)^2 = \dfrac{\Lambda}{3}$, OR $\dot a = \sqrt{\dfrac{\Lambda}{3}}\,a$.

AS IN PROB. 5.3, THE EXPANSIVE SOLUTION OF THIS O.D.E. IS $a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}}t}$. SINCE $q = \dfrac{\Omega}{2} - \Omega_\Lambda$, $\boxed{\text{AS } t \to \infty,\ q \to -1.}$

7.8

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \qquad \text{FROM } (6.7)$$

FOR MATTER-DOMINATED UNIVERSE

$$\rho(t) = \rho_0 / a^3 \qquad (5.15)$$

FOR A FLAT UNIVERSE WITH MATTER AND Λ

$$\rho_c(t) = \rho(t) + \rho_\Lambda$$

NOTE, ρ_Λ CONSTANT BY EQN (7.6)

SO,

$$\Omega(t) = \frac{\rho_0/a^3}{\rho_0/a^3 + \rho_\Lambda}$$

AT CURRENT TIME, t_0,

$$\Omega_0 = \rho_0 / \rho_c(t_0) \quad \text{AND } \rho_c(t_0) = \rho_0 + \rho_\Lambda, \text{ SO}$$

$$= \rho_0 / (\rho_0 + \rho_\Lambda) \Rightarrow \rho_\Lambda = \rho_0 \left(\frac{1}{\Omega_0} - 1 \right)$$

THUS $\Omega(t) = \dfrac{\rho_0/a^3}{(\rho_0/a^3) + \rho_0\left(\frac{1}{\Omega_0} - 1\right)}$

MULTIPLY TOP & BOTTOM BY $1/\rho_c = \Omega_0/\rho_0$ AND

$$\Omega(t) = \frac{\Omega_0/a^3(t)}{1 - \Omega_0 + \Omega_0/a^3}$$

BUT, BY (5.10) WITH $a(t_r) = a(t_0) = 1$, $a = 1/(1+z)$, SO

$$\boxed{\Omega(t) = \Omega_0 \frac{(1+z)^3}{1 - \Omega_0 + (1+z)^3 \Omega_0}}$$