INTRO TO MODERN COSMOLOGY, 2nd ED.
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PROBLEM SOLUTIONS BY BILL DANIEL

CHAPTER 2

2.1 $M_{GAL} \cong 10^{11} M_{\odot} \cong 10^{11} \times 2\times10^{30}\,Kg = 2\times10^{41}\,Kg$

$V_{GAL} \cong (1\,Mpc)^3 \cong (3\times10^{16}\times10^{6}\,m)^3 = 3\times10^{67}\,m^3$

$\rho_{UNIV} = M_{GAL}/V_{GAL} = 2\times10^{41}\,Kg/3\times10^{67}\,m^3 = \boxed{7\times10^{-27}\,Kg/m^3}$

$M_{EARTH} \cong 6\times10^{24}\,Kg$

$V_{EARTH} \cong \frac{4}{3}\pi(6.4\times10^{6}\,m)^3 = 1.7\times10^{20}\,m^3$

$\rho_{EARTH} = M_{EARTH}/V_{EARTH} = 6\times10^{24}\,Kg/1.7\times10^{20}\,m^3 = \boxed{3.5\times10^{4}\,kg/m^3}$

$\boxed{\rho_{EARTH}/\rho_{UNIV} = 5\times10^{30}}$

2.2 BECAUSE THE PECULIAR VELOCITIES ARE RANDOMLY ORIENTED, $v^2 = v_x^2 + v_y^2 + v_z^2$ AND $v_x = v_y = v_z$. THUS THE SQUARE OF THE RADIAL PECULIAR VELOCITY IS 1/3 OF THE SQUARE OF THE RANDOMLY ORIENTED PECULIAR VELOCITY.

$$v_{RAD}^{PEC} = \sqrt{\frac{600^2}{3}}\,km/s = 350\,km/s$$

HUBBLE'S LAW STATES THAT $H_0 = v_{RAD}/r$. IF WE CAN TOLERATE AN ERROR OF 10% IN H_0 AND WE ASSUME THAT ALL UNCERTAINTY IN v_{RAD} IS DUE TO THE PECULIAR

VELOCITY, v_{RAD}^{PEC}, THEN $r \geq \frac{v_{RAD}^{PEC}}{0.10} / H_0$.

a) $H_0 = 100\left(\frac{km}{s}\right)/Mpc$: $r \geq \left(\frac{350}{0.10}\right)/100 = \boxed{35\,Mpc}$

b) $H_0 = 50\left(\frac{km}{s}\right)/Mpc$: $r \geq \left(\frac{350}{0.10}\right)/50 = \boxed{70\,Mpc}$

2.3 IF THERE WERE A NET CHARGE, ITS MOTION WOULD GENERATE AN OVERALL MAGNETIC FIELD THAT WE DO NOT OBSERVE.

2.4 $f = E/h = \frac{(13.6\,eV)(1.6 \times 10^{-19}\,J/eV)}{6.6 \times 10^{-34}\,J \cdot s} = \boxed{3.3 \times 10^{15}\,Hz}$

$T \cong E/3k_B = \frac{(13.6\,eV)(1.6 \times 10^{-19}\,J/eV)}{(3)(1.4 \times 10^{-23}\,J/K)} = \boxed{5.2 \times 10^{5}\,K}$

2.5 $f_{PEAK} \cong 2.8\,k_B T/h$. SINCE k_B AND h ARE CONSTANTS, SO IS $f_{PEAK}/T \cong 2.8\,k_B/h$.

$\frac{f_{PEAK}}{T} \cong \frac{(2.8)(1.38 \times 10^{-23}\,J/K)}{6.6 \times 10^{-34}\,J \cdot s} = \boxed{5.8 \times 10^{10}\,Hz/K}$

$f_{PEAK\odot} \cong (5.8 \times 10^{10}\,Hz/K)(5800\,K) = \boxed{3.4 \times 10^{14}\,Hz}$

2.6 $f_{PEAK,CMB} \cong (5.8 \times 10^{10}\,Hz/K)(2.725\,K) = 1.6 \times 10^{11}\,Hz$

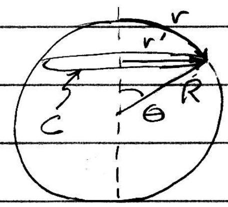
$\lambda_{PEAK,CMB} = c/f_{PEAK,CMB} \cong \frac{3 \times 10^{8}\,m/s}{1.6 \times 10^{11}\,s^{-1}} = \boxed{1.9 \times 10^{-3}\,m}$

BY (2.10) AND (2.11): $\varepsilon_{RAD} = \alpha T^4 = (7.565 \times 10^{-16}\,J/m^3K^4)(2.725\,K)^4$

$= \boxed{4.17 \times 10^{-14}\,J/m^3}$

CHAPTER 4

4.1



$\sin\Theta = r'/R$

RADIAN MEASURE:

$$\frac{r}{2\pi R} = \frac{\Theta}{2\pi} \Rightarrow \Theta = r/R$$

$$C = 2\pi r' = 2\pi R \sin\Theta = \boxed{2\pi R \sin(r/R)}$$

CHECKS:

AT $\Theta = \emptyset$; $C = 2\pi R \sin(\emptyset) = \emptyset$ ✓

AT $\Theta = 90°$; $C = 2\pi R \sin(\pi/2) = 2\pi R$ ✓

FOR $r \ll R$, $\sin(r/R) \to r/R$, SO

$C = 2\pi R(r/R) = \boxed{2\pi r\text{, AS EXPECTED IN A FLAT GEOMETRY.}}$

4.2 THE AREA OF A CIRCLE IN THE SPHERICAL GEOMETRY ABOVE IS

$$A_s = \int_0^r C\,dr = 2\pi R \int_0^r \sin\left(\frac{r''}{R}\right)dr'' \quad \left(\text{let } u = \frac{r''}{R};\, du = \frac{dr''}{R}\right)$$

$$= 2\pi R^2\left(-\cos\left(\frac{r''}{R}\right)\right)\Big|_0^r$$

$$= 2\pi R^2\left(1 - \cos\left(\frac{r}{R}\right)\right)$$

$N_s = nA_s = \boxed{2\pi n R^2\left(1 - \cos\left(\frac{r}{R}\right)\right)}$, TAYLOR EXPANDING,

$$N_s = 2\pi n R^2\left(1 - \left(1 - \frac{r^2}{2R^2} + \ldots\right)\right)$$

$$N_F \cong (2\pi n R^2)\left(\frac{r^2}{2R^2}\right) = \boxed{\pi n r^2}$$

4.2 cont. So, $N_S = 2\pi n R^2(1-\cos(\frac{r}{R}))$ AND $N_F = n\pi r^2$

$$\frac{N_S}{N_F} = \frac{2R^2(1-\cos(\frac{r}{R}))}{r^2} = \frac{2(1-\cos\theta)}{\theta^2}$$

TO SHOW THE RELATIONSHIP BETWEEN N_S AND N_F REDUCES TO SHOWING THE RELATIONSHIP BETWEEN $2(1-\cos\theta)$ AND θ^2.
OBVIOUSLY THEY ARE BOTH $\emptyset$ AT $\theta=\emptyset$. TO INVESTIGATE THEIR BEHAVIOR FOR $\theta>\emptyset$, TAKE THEIR DERIVATIVES:

$$\frac{d}{d\theta}(2(1-\cos\theta)) = 2\sin\theta \qquad \frac{d}{d\theta}\theta^2 = 2\theta$$

SINCE $\sin\theta < \theta$ FOR ALL $\theta>\emptyset$, $N_S < N_F$ FOR ALL $r>0$, AND WE SEE FEWER GALAXIES IN A SPHERICAL UNIVERSE.

4.3 FOR A GIVEN UNIVERSE, K IS FIXED, SO $K>0$ CAN NEVER BECOME $K<0$ AND VICE VERSA, SO THE OPEN/CLOSED STATUS OF A UNIVERSE CAN NEVER CHANGE.

CHAPTER 5

5.1 THE UNIVERSE CAN NEVER LOSE/GAIN ENERGY AS THERE IS NOWHERE ELSE FOR IT TO GO/COME FROM.

5.2 (2.4) $E^2 = m^2c^4 + p^2c^2$, BUT FOR PHOTONS, $m=0$, SO $E^2 = p^2c^2 \rightarrow E = pc$ OR $p = E/c$

FOR PHOTON $v = c$, SO $\vec{v}\cdot\vec{p} = (c)(E/c) = E$

SO,

$$P = \frac{1}{3} n \langle \vec{v}\cdot\vec{p} \rangle = \boxed{\frac{1}{3} n \langle E \rangle}$$

5.3 AS ALWAYS, BEGIN WITH THE FLUID EQUATION

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

SUBSTITUTE: $P = (\gamma - 1)\rho c^2$ TO GET

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\gamma\rho) = 0$$

THIS IS THE SAME AS THE MATTER-ONLY CASE WITH $3 \rightarrow 3\gamma$, SO $\boxed{\rho(a) = \rho_0 / a^{3\gamma}}$

THE FRIEDMANN EQUATION WITH $k=0$ THEN IS

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\frac{\rho_0}{a^{3\gamma}}\right)$$

THUS

$$\dot{a}^2 = \frac{8\pi G}{3} \cdot \rho_0 a^{2-3\gamma}$$

ASSUMING A POWER LAW SOLUTION, $a \propto t^q \Rightarrow \dot{a} \propto t^{q-1}$ AND EQUATING POWERS GIVES $2(q-1) = q(2-3\gamma)$, OR $q = 2/3\gamma$.

(FOR MATTER, $\gamma = 1 \Rightarrow q = 2/3$; FOR RADIATION, $\gamma = 4/3 \Rightarrow q = 1/2$.)

5.3 Cont. THUS $\boxed{a(t) = \left(\frac{t}{t_0}\right)^{2/3\gamma}}$; $\boxed{\rho(t) = \frac{\rho_0}{a^{3\gamma}} = \frac{\rho_0 t_0^2}{t^2}}$

IF $P = -\rho c^2$ THEN $\gamma = 0$, q BLOWS UP AND THIS APPROACH GIVES NO SOLUTION. INSTEAD, RETURN TO $\rho = \rho_0 / a^{3\gamma}$. HAVING $\gamma = 0$ MEANS $\rho = \rho_0$ AND $\dot{a}^2 = \frac{8\pi G}{3}\rho_0 a^2$, HENCE,

$\dot{a} = \pm\sqrt{\frac{8\pi G \rho_0}{3}}\, a$, A FIRST ORDER O.D.E. WITH SOLUTION $\boxed{a(t) \propto \exp\left(\pm\sqrt{\frac{8\pi G\rho_0}{3}}\, t\right)}$.

5.4 $a(t) = (t/t_0)^{2/3\gamma}$; $\rho \propto (t/t_0)^{-2}$

SINCE K IS A CONSTANT, K/a^2 HAS THE SAME TIME DEPENDENCE AS a^{-2}, THAT IS, $-4/3\gamma$. THUS, IF ρ AND K/a^2 HAVE THE SAME TIME DEPENDENCE,

$$-4/3\gamma = -2 \Rightarrow \boxed{\gamma = 2/3}$$

THE FULL FRIEDMANN EQN,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$

WITH $\gamma = 2/3$ (AND HENCE $\rho = \rho_0/a^{3\gamma} = \rho_0/a^2$)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^2} - \frac{K}{a^2} \Rightarrow$$

$$\dot{a}^2 = \frac{8\pi G \rho_0}{3} - K.$$

SO, WITH $K < 0$, $\dot{a}^2 > 0 \Rightarrow \dot{a}$ IS A REAL CONSTANT AND $\boxed{a \propto t}$

5.5 WITH $k>0$; $p=0$; $\rho=\rho_0/a^3$, TAKE $\beta=\frac{4\pi G\rho_0}{3}$.

THUS:

$$a(\theta)=\frac{\beta}{k}(1-\cos\theta) \text{ AND } t(\theta)=\frac{\beta}{k^{3/2}}(\theta-\sin\theta)$$

$$\dot{a}=\frac{da}{dt}=\frac{da}{d\theta}\cdot\left(\frac{dt}{d\theta}\right)^{-1}$$

$$\frac{da}{d\theta}=\frac{\beta}{k}\sin\theta \; ; \; \frac{dt}{d\theta}=\frac{\beta}{k^{3/2}}(1-\cos\theta)$$

$$\frac{da}{dt}=\left(\frac{\beta}{k}\sin\theta\right)\left(\frac{k^{3/2}}{\beta(1-\cos\theta)}\right)=\frac{\sqrt{k}\sin\theta}{(1-\cos\theta)}$$

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{k\sin^2\theta}{(1-\cos\theta)^2}\cdot\frac{k^2}{\beta^2(1-\cos\theta)^2}=\frac{k^3\sin^2\theta}{\beta^2(1-\cos\theta)^4}$$

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G\rho}{3}-\frac{k}{a^2}=\frac{8\pi G\rho_0}{3a^3}-\frac{k}{a^2}=\frac{2\beta-ka}{a^3}$$

$$=\frac{2\beta-k\left(\frac{\beta}{k}(1-\cos\theta)\right)}{\left(\frac{\beta}{k}(1-\cos\theta)\right)^3}=\frac{k^3\beta\,(1+\cos\theta)}{\beta^3\,(1-\cos\theta)^3}\cdot\frac{(1-\cos\theta)}{(1-\cos\theta)}$$

$$=\frac{k^3\sin^2\theta}{\beta^2(1-\cos\theta)^4} \checkmark$$

THIS UNIVERSE OSCILLATES WITH A PERIOD OF

$\tau=\frac{8\pi^2 G\rho_0}{3k^{3/2}}$. ITS MAXIMUM SIZE IS

$(a(\theta)=0$ FOR $\theta=2n\pi$. $t(2n\pi)=2n\pi\beta/k^{3/2}$.)

$$a_{MAX}=\frac{8\pi G\rho_0}{3k}.$$

(a_{MAX} OCCURS AT $\theta=\pi$, SO $a(\pi)=a_{MAX}=2\beta/k$.)

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Plot[1 - Cos[θ], {θ, 0, 2 Pi}, AxesLabel → {"β/k", "a(θ)=β/k (1-cos θ)"}]
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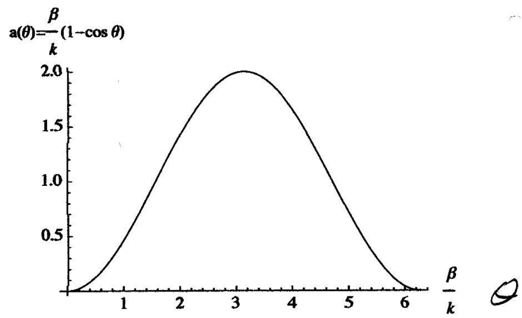


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Plot[θ - Sin[θ], {θ, 0, 2 Pi}, AxesLabel → {"β/k^(3/2)", "t(θ)=β/k^(3/2) (θ-sin θ)"}]
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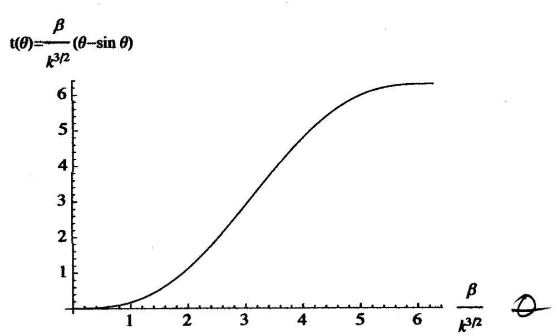

Below is a plot of $a(t)$ parameterized by θ. As in the plot above of $a(\theta)$, the scale factor reaches a maximum of $a_{max} = \frac{2\beta}{k}$. This occurs at $t = \frac{\pi\beta}{k^{3/2}}$.

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ParametricPlot[{θ - Sin[θ], 1 - Cos[θ]}, {θ, 0, 2 Pi}, AxesLabel → {"β/k^(3/2)", "β/k"}]
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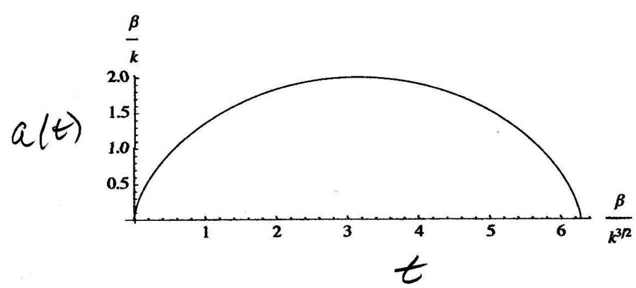

5.6 $k < 0;\ p = 0;\ \rho = \frac{\rho_0}{a^3}$

IF LAST TERM OF FRIEDMANN EQN DOMINATES,

$\dot{a} = \sqrt{|k|} \Rightarrow a = \sqrt{|k|}\, t + a_0 = \sqrt{|k|}\, t + 1$

$a(t) = \sqrt{|k|}\, t + 1$, $\boxed{a \,\tilde{\alpha}\, t}$

$\rho(t) = \rho_0 / (\sqrt{|k|}\, t + 1)^3$ SO $\boxed{\rho(t) \,\tilde{\alpha}\, t^{-3}}$

THIS IS A THINNING UNIVERSE THAT NEVER COLLAPSES. IT IS STABLE.