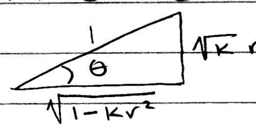
ADVANCED TOPIC 2 PROBLEMS

AT2.1 EVEN THOUGH THE EXPANSION SLOWS TO A STOP, THE UNIVERSE WAS STILL EXPANDING, SO THE LIGHT IS REDSHIFTED.

AT2.2 THIS INTEGRAL IS EVALUATED USING A TRIG. SUB.

LET $\theta = \sin^{-1}(\sqrt{K}r)$; $-\pi/2 \le \theta \le \pi/2$.



$$\frac{d\theta}{dr} = \frac{d\sin^{-1}(\sqrt{K}r)}{dr} = \left(\frac{K}{1-Kr^2}\right)^{1/2}$$

$$dr = \left(\frac{1-Kr^2}{K}\right)^{1/2} d\theta$$

SO,

$$d_{PHYS} = a_0 \int_0^{\sin^{-1}(\sqrt{K}r_0)} \left(\frac{1-Kr^2}{K}\right)^{1/2}\left(\frac{1}{1-Kr^2}\right)^{1/2} d\theta$$

$$= \frac{a_0}{\sqrt{K}} \int_0^{\sin^{-1}(\sqrt{K}r_0)} d\theta = \boxed{\frac{a_0}{\sqrt{K}} \sin^{-1}(\sqrt{K}r_0)}$$

BY (A2.15) $d_{lum} = a_0 r_0 (1+z)$, BUT WE NEED THIS IN TERMS OF d_{PHYS}, SO WRITE

$$r_0 = \frac{1}{\sqrt{K}} \sin\left(\frac{\sqrt{K}}{a_0} d_{PHYS}\right)$$

AND

$$\boxed{d_{lum} = a_0 r_0 (1+z) = \frac{a_0}{\sqrt{K}}(1+z)\sin\left(\frac{\sqrt{K}}{a_0} d_{PHYS}\right)}$$

FOR SMALL α, $\sin\alpha \cong \alpha$, SO

$$d_{lum} \cong \frac{a_0}{\sqrt{K}}(1+z)\left(\frac{\sqrt{K}}{a_0} d_{PHYS}\right) = \boxed{(1+z)d_{PHYS}}$$

FOR NEARBY OBJECTS $z \approx \emptyset$ SO $\boxed{d_{lum} \cong d_{PHYS}}$.

d_{PHYS} HAS NO z DEPENDENCE, SO, FOR DISTANT OBJECTS, AS EXPECTED, IT IS UNAFFECTED BY REDSHIFT.

A2.2 CONT. BUT d_{lum} INCREASES WITH DISTANCE BECAUSE z DOES. COUNTERING THIS EFFECT IS THE FACTOR OF $\sin$ THAT IS ALWAYS ≤ 1.

A2.3 BY (A2.4) $\displaystyle\int_{t_0}^{t_e} \frac{c\,dt}{a(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}$.

WE JUST SOLVED THE RHS INTEGRAL IN PROB. A2.2 TO GET: $\frac{1}{\sqrt{k}} \sin^{-1}(\sqrt{k}\, r_0)$.

BUT, FOR $\lim_{k \to 0} \sin^{-1}(\sqrt{k}\, r_0) = \sqrt{k}\, r_0$, SO RHS $= \frac{\sqrt{k}\, r_0}{\sqrt{k}} = r_0$

THE LHS IS INTEGRATED BY RECALLING (5.15)

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}, \text{ SO}$$

$$r_0 = c\, t_0^{2/3} \int_{t_e}^{t_0} t^{-2/3}\, dt = 3c t_0^{2/3} \left(t_0^{1/3} - t_e^{1/3}\right)$$

$$= 3ct_0 \left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right] = 3ct_0 \left[1 - a(t_e)^{1/2}\right]$$

BY (A2.10) AND $a(t_0) = 1$

$$\boxed{r_0 = 3ct_0 \left[1 - 1/\sqrt{1+z}\right]}$$

BY (A2.19) $\displaystyle \theta = \frac{\ell(1+z)}{a_0 r_0} = \frac{\ell}{3ct_0} \cdot \frac{(1+z)}{1 - 1/\sqrt{1+z}}$

$$\boxed{\theta = \frac{\ell}{3ct_0} \cdot \frac{(1+z)^{3/2}}{(1+z)^{1/2} - 1}}$$

FOR SMALL z, THE NUMERATOR OF THE z FRACTION GOES TO 1, THE DENOMINATOR, BY TAYLOR EXPANSION $(1+z)^{1/2} - 1 \approx \frac{1}{4}z - \frac{1}{24}z^2 + \cdots \propto z$ FOR SMALL z, SO $\theta \propto 1/z$ FOR SMALL z.

A2.3 cont.

FOR LARGE z, THE 1'S IN BOTH NUMERATOR AND DENOMINATOR ARE INSIGNIFICANT AND

$$\Theta \propto \frac{z^{3/2}}{z^{1/2}} = z.$$

NEARBY OBJECTS GET SMALLER WITH DISTANCE AS $1/z$, JUST AS WE WOULD EXPECT. DISTANT OBJECTS, THOUGH GET LARGER WITH DISTANCE! THIS IS DUE TO THE FACT THAT LARGE z MEANS EARLIER TIME, AND AT THUS SIZES OF OBJECTS AT EARLIER TIMES APPEAR LARGER BECAUSE THEIR SURROUNDING SPACETIME HAS NOT EXPANDED MUCH.

A2.4

TO FIND THE DISTANCE AT WHICH AN OBJECT APPEARS SMALLEST, MINIMIZE $\Theta(z)$.

$$\frac{d\theta}{dz} = \frac{l}{3ct_0} = \frac{[\sqrt{1+z}-1][\frac{3}{2}(1+z)^{1/2}] - (1+z)^{3/2}(\frac{1}{2}(1+z)^{-1/2})}{(1+z) - 2(1+z)^{1/2} + 1} = 0$$

$$\Rightarrow \frac{3}{2}(1+z) - \frac{3}{2}(1+z)^{1/2} - \frac{1}{2}(1+z) = 0$$

$$1+z = \left(\frac{3}{2}\right)^2 \Rightarrow \boxed{z = 5/4}$$

A2.4

$$H^2(t) = \frac{8\pi G}{3}(\rho + \rho_\Lambda) \qquad \text{FROM (7.7) WITH } K=0$$

$\rho_\Lambda = \rho_0\left(\frac{1}{\Omega_0} - 1\right)$ FROM SOLUTION TO PROBLEM 7.5

$\rho = \rho_0/a^3$ FOR MATTER-DOMINATED UNIVERSE (5.15)

SO,

$$H^2(t) = \frac{8\pi G}{3}\left(\frac{\rho_0}{a^3} + \rho_0\left(\frac{1}{\Omega_0} - 1\right)\right) = \frac{8\pi G}{3}\rho_0\left(\frac{1}{a^3} + \frac{1}{\Omega_0} - 1\right)$$

$$= \frac{8\pi G}{3}\frac{\rho_0}{\Omega_0}\left(\frac{\Omega_0}{a^3} + 1 - \Omega_0\right)$$

A2.4 cont.

BUT $\Omega_0 = \rho_0 / \rho_c(t_0) = \rho_0 / \rho_0 + \rho_\Lambda$ AND

$a = 1/(1+z)$ FROM (5.10), SO

$$H^2(t) = \frac{8\pi G}{3}(\rho_0+\rho_\Lambda)\left(1-\Omega_0+\Omega_0(1+z)^3\right)$$

BUT, BY (7.7)

$$H_0^2 = \frac{8\pi G}{3}(\rho_0+\rho_\Lambda) \quad , \text{SO}$$

$$\boxed{H^2(t) = H_0^2\left(1-\Omega_0+\Omega_0(1+z)^3\right)}$$

A2.5

THE FLUX LIMIT, S, DETERMINES THE MAXIMUM DISTANCE, r_0, AT WHICH SOURCES OF LUMINOSITY, L, CAN BE SEEN. IF WE TAKE $r_0 = d_{lum} = (L/S)^{1/2}$, THEN $N \propto r^3 \propto S^{-3/2}$. IF THE SOURCES HAVE DIFFERENT LUMINOSITIES, BUT ARE EVENLY DISTRIBUTED, EACH POPULATION WITH LUMINOSITIES BETWEEN L AND L+dL WILL EXHIBIT THE SAME $N \propto S^{-3/2}$ SCALING PROPERTY. THUS, SO WILL A POPULATION COMPOSED OF SOURCES WITH A RANGE OF LUMINOSITIES. SINCE $\frac{dN}{dS} \propto -S^{-5/2}$, N DECREASES RAPIDLY WITH DECREASING S, SO MOST SOURCES WILL BE FOUND NEAR THE FLUX LIMIT.

A2.4 Cont.

BY (A2.10) WITH $a(t_0)=1$

$$\frac{1}{a(t)} = 1+z$$

TAKING THE TIME DERIVATIVE OF BOTH SIDES,

$$\frac{-1}{a^2(t)}\dot{a} = \frac{dz}{dt}, \text{ OR}$$

$$-\frac{H}{a} = \frac{dz}{dt}, \text{ OR } -\frac{dz}{H} = \frac{dt}{a}$$

NOW, FROM (A2.12) AND (5.15)

$$r_0 = c\, t_0^{2/3}\int_0^{t_0}\frac{dt}{t^{2/3}} = c\int_0^{t_0}\frac{dt}{a} = c\int_0^{z}\frac{dz}{H}$$

(INTEGRATION LIMIT FROM $-t_0 \to z$)

USING THE FIRST EXPRESSION IN PROBLEM A2.4

$$\boxed{r_0 = cH_0^{-1}\int_0^{z}\frac{dz'}{[1-\Omega_0+\Omega_0(1+z')^3]^{1/2}}}$$

TAKING $\Omega_0=1$, $cH_0^{-1}=3000h^{-1}$ Mpc, $a_0=1$ AND USING (A2.15) AND (A2.20):

$$d_{lum} = a_0 r_0(1+z) = 3000h^{-1}(1+z)\int_0^{z}\frac{dz'}{(1+z')^{3/2}} = 6000h^{-1}\left[1+z-(1+z)^{1/2}\right]$$

$$d_{diam} = \frac{d_{lum}}{(1+z)^2} = 6000h^{-1}\left[\frac{1+z-(1+z)^{1/2}}{(1+z)^2}\right]$$

THE INTEGRAL IN THE BOX ABOVE CAN'T BE SOLVED IN CLOSED FORM FOR $\Omega_0 \neq 1$, SO THE OTHER CURVES IN FIGURES A2.3 AND A2.5 ARE THE RESULT OF NUMERICAL INTEGRATION.