DERIVATION OF EQUATIONS IN THE LIDDLE TEXT (2^{nd} ED)

(2.2) FOR SMALL v/c, $\lambda_{obs} = (1+v/c)\lambda_{em}$. SO (2.2) COMES FROM (2.1) AS:

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{(1+\frac{v}{c})\lambda_{em} - \lambda_{em}}{\lambda_{em}} = \frac{v}{c}$$

(2.5) THE FIRST EQUALITY IS JUST AN ALGEBRAIC REARRANGEMENT OF (2.4). THE SECOND (APPROXIMATE) EQUALITY IS A MACLAURIN SERIES EXPANSION OF THIS RESULT. RECALL THAT FOR SMALL p (LOW VELOCITY) THE MACLAURIN THEOREM SAYS:

$$f(p) = f(0) + p f'(0) + \frac{1}{2!}p^2 f''(0) + \cdots$$

WHERE $f^{(n)}(0)$ MEANS TO DIFFERENTIATE n TIMES W.R.T. p AND THEN SET $p=0$. SO, HERE:

$$f(p) = (1 + p^2/m^2c^2)^{1/2}$$

$$f(0) = 1$$

$$f'(p) = \frac{1}{2}(1 + p^2/m^2c^2)^{-1/2}(2p/m^2c^2) = \frac{p}{m^2c^2}(1 + p^2/m^2c^2)^{-1/2}$$

$$f'(0) = 0$$

$$f''(p) = 1/m^2c^2(1 + p^2/m^2c^2)^{-1/2} + \frac{p}{m^2c^2}\left(-\frac{1}{2}\right)(1 + p^2/m^2c^2)^{-3/2}\left(\frac{2p}{m^2c^2}\right)$$

$$f''(0) = 1/m^2c^2$$

SO,

$$f(p) = f(0) + p f'(0) + \frac{1}{2!}p^2 f''(0) + \cdots$$

$$\left(1+\frac{p^2}{m^2c^2}\right)^{1/2} = 1 + p(0) + \frac{1}{2}p^2\left(\frac{1}{m^2c^2}\right) + \cdots$$

$$\simeq 1 + \frac{1}{2}\frac{p^2}{m^2c^2}$$

(2.9) $$\mathcal{E}_{rad} = \int_0^\infty \mathcal{E}(f)\,df = \frac{8\pi h}{c^3}\int_0^\infty \frac{f^3\,df}{\exp(hf/K_BT)-1}$$

let $y = \dfrac{hf}{K_BT} \rightarrow f = \dfrac{K_BTy}{h} \rightarrow f^3 = \dfrac{K_B^3T^3y^3}{h^3}$

$$\frac{dy}{df} = \frac{h}{K_BT} \longrightarrow df = \frac{K_BT\,dy}{h}$$

$$\mathcal{E}_{rad} = \left(\frac{8\pi h}{c^3}\right)\left(\frac{K_B^3T^3}{h^3}\right)\left(\frac{K_BT}{h}\right)\int_0^\infty \frac{y^3dy}{e^y-1}$$

$$= \frac{8\pi K_B^4}{c^3h^3}T^4\int_0^\infty \frac{y^3dy}{e^y-1}$$

(2.10) THIS INTEGRAL IS A SPECIAL CASE ($z=1$, $s=4$) OF THE POLYLOGARITHM FUNCTION, $Li_s(z)$. THE EASIEST WAY (THAT I KNOW OF) TO EVALUATE THIS INTEGRAL IS TO RECOGNIZE THAT THE POLYLOGARITHM CAN BE REPRESENTED AS A POWER SERIES:

$$Li_s(z) = \sum_{K=1}^{\infty}\frac{z^K}{K^s} = \frac{1}{\Gamma(s)}\int_0^\infty \frac{y^{(s-1)}dy}{e^{y/z}-1}.$$

THE SERIES IS AN EXAMPLE OF THE RIEMANN ZETA FUNCTION, $\zeta(s) = \sum_{K=1}^{\infty} 1/K^s$ (THE MOST FAMOUS BEING THE EULER RESULT FOR $s=2$, $\sum_{K=1}^{\infty} 1/K^2 = \pi^2/6$!).

THE GENERAL FORMULA FOR ZETA FUNCTIONS IS:

$$\zeta(2n) = \sum_{K=1}^{\infty} 1/K^{2n} = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2\cdot(2n)!},$$

(2.10) Cont. WHERE B_j IS THE j^{th} BERNOULLI NUMBER. FOR OUR CASE $B_4 = -1/30$, SO $\zeta(4) = -\frac{(-1/30)(2\pi)^4}{2(4!)} = \pi^4/90$.

THE GAMMA FUNCTION IS A GENERALIZATION OF THE FACTORIAL FUNCTION, WITH $\Gamma(n) = (n-1)!$, SO $1/\Gamma(4) = 1/6$. OUR INTEGRAL IS THEN:

$$\int_0^\infty \frac{y^3 dy}{e^y - 1} = \frac{\pi^4}{90} \cdot 6 = \frac{\pi^4}{15}.$$

THUS, $$\mathcal{E}_{rad} = \frac{8\pi k_B^4}{h^3 c^3} T^4 \int_0^\infty \frac{y^3 dy}{e^y - 1}$$

$$= \frac{8\pi k_B^4}{h^3 c^3} \cdot \frac{\pi^4}{15} T^4 = \frac{1}{15} \cdot \frac{\pi^2 k_B^4}{c^3} \cdot \frac{8\pi^3}{h^3} \cdot T^4$$

WITH $\hbar = \frac{h}{2\pi}$,

$$= \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} \cdot T^4 = \alpha T^4 \quad \text{WITH } \alpha = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$$

(3.9) EQUATION (3.8) GIVES THE RELATIONSHIP BETWEEN THE PHYSICAL DISTANCE, $\vec{r}$, AND THE COMOVING DISTANCE, $\vec{x}$:

$$\vec{r} = a(t)\vec{x}$$

TO CONVERT (3.7) INTO (3.9), WE NEED $\dot{\vec{r}}$.

$$\dot{\vec{r}} = \dot{a}\vec{x} + a\dot{\vec{x}}$$

BUT, $\dot{\vec{x}} = 0$, SINCE DISTANCES TO FIXED POINTS IN THE COMOVING FRAME DO NOT CHANGE, SO $\dot{\vec{r}} = \dot{a}\vec{x}$, AND SIMPLE SUBSTITUTION GIVES (3.9).

(3.15) THE MASS-ENERGY OF THE CONTENTS OF A SPHERICAL VOLUME, V, CONTAINING A MASS DENSITY ρ IS:

$$E = mc^2 = V\rho c^2 = \frac{4}{3}\pi a^3 \rho c^2 \qquad (3.12)$$

BOTH a AND ρ ARE FUNCTIONS OF TIME, SO,

$$\frac{dE}{dt} = \frac{4}{3}\pi \rho c^2 (3a^2)\dot{a} + \frac{4}{3}\pi a^3 c^2 \dot{\rho}$$

$$= 4\pi \rho a^2 c^2 \dot{a} + \frac{4}{3}\pi a^3 \dot{\rho} c^2. \qquad (3.13)$$

SINCE THE VOLUME OF THE SPHERE IN COMOVING COORDINATES IS: $V = \frac{4}{3}\pi a(t)^3$,

$$\frac{dV}{dt} = \frac{4}{3}\pi (3a^2\dot{a}) = 4\pi a^2 \dot{a}. \qquad (3.14)$$

REVERSIBLE ADIABATIC EXPANSIONS ARE ISENTROPIC ($ds=0$), SO,

$$\overbrace{4\pi\rho a^2 c^2 \dot{a} + \frac{4}{3}\pi a^3 \dot{\rho} c^2}^{dE/dt} + p\,(\overbrace{4\pi a^2 \dot{a}}^{dV/dt}) = 0.$$

REARRANGING GIVES:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0. \qquad (3.15)$$

(3.18) DERIVATION OF (3.16-18) IS WORTH DOING, BUT IS STRAIGHTFORWARD.

(5.10) EQN. (5.6) FOLLOWS FROM DIFFERENTIATION OF $\vec{v} = H_0 \vec{r}$ (2.3) AND RECOGNIZING THAT IT APPLIES AT ALL TIMES (SO $H_0 \rightarrow H$). IF WE DEFINE $d\lambda \equiv \lambda_r - \lambda_e$ WE CAN WRITE EQN. (2.1) (WITH $\lambda_{OBS} \rightarrow \lambda_r$ AND $\lambda_{em} \rightarrow \lambda_e$) AS: $z = \dfrac{d\lambda}{\lambda_e}$, THIS IS ALSO $z = dv/c$ FROM (2.2) WRITING dv FOR v TO EMPHASIZE THAT RECESSION IS RELATIVE TO A MOVING EARTH.

EQN. (5.8) FOLLOWS EASILY BY RECOGNIZING THAT $\dot{a} = da/dt$ IN THE LAST EQUALITY. AND EQN. (5.10) FOLLOWS FROM (2.1): $z = (\lambda_r - \lambda_e)/\lambda_e = -1 + \lambda_r/\lambda_e$. SO, $1 + z = \lambda_r/\lambda_e$, AND, BY (5.9) $1 + z = a(t_r)/a(t_e)$.