LEARNING QUANTUM MECHANICS AND RELATIVITY

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Abstract. Students of quantum mechanics and relativity encounter material that is not only mathematically difficult but also conceptually incredulous. They ask, "Isn't there a way to make the mathematics and its interpretation more transparent?" and also, "Is this the way the world really is?" Standard texts on non-relativistic quantum mechanics (QM) generally focus on the abstract mathematical machinery for solving problems and usually minimize or avoid attempts at intuitive understanding, basic underlying physics, and the existence of many differing interpretations of the mathematics. Also, different texts present quite different sets of fundamental but abstract postulates for a coherent system from which to calculate probabilities of experimental outcomes. The goal of this paper is to balance out that strangeness and abstraction by providing an intuitive understanding of some of the key parts of this machinery. We wish to motivate and simplify so that the mathematics isn't quite so formidable. After this introduction, standard texts may be studied in the usual way.

Special and general relativity also have their abstractness and opaqueness. Why, for example, do their metrics have differences in sign between space and time parts? And, when dealing with weak gravitational fields, is it still necessary to know the language of tensor calculus or advanced differential geometry?

Most of the special heuristic tidbits discussed below are not well known. Why that should be is largely a mystery to me.

1. Introduction

The mathematical theory of quantum mechanics is highly successful and has flawlessly passed nearly ninety years of careful experimental tests. College textbooks on quantum mechanics generally do a good job of providing adequate coverage of topic material so that students have a conventional common mathematical machinery for solving relevant physics problems. But this is often done in a sparse fashion which presents abstract postulates and rules without sufficient motivation or physical clarity. They don't say why we do things in this conventional semi-Copenhagen way, how much linear algebra one should have first and why, what's really going on, where's the physics beneath the abstract mathematics? They presume that the machinery will make some sense (or at least familiarity) after solving a series of problems. But basic postulates and math are counterintuitive and are given "out of the blue;" and application is done from abstract generals to particular examples. It is fairly easy to claim that in quantum mechanics, heuristics are poor. But there is also a reason for this: physicists do not agree about the possibilities for any underlying reality.

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Some even claim that there is no underlying reality. And there is no general agreement on the interpretation of the mathematics of quantum mechanics. As an example, one reason that Niels Bohr was so difficult to understand was that he was very careful to always avoid any mention of possible underlying mechanisms. So, if you are one of those people who ask, "What is really going on?," you may find few answers.

Every text on quantum mechanics provides a list of postulates from which basic structure can be developed and problems solved for the probabilities of experimental outcomes. Postulates are often stated with numbers (like [P1]) with orderings and main choices that vary from text to text. Here, I will assign numbers to primary postulates, but the choice is fairly arbitrary. Postulate 1 [P1] of quantum mechanics is always about the existence of a complex state function, $\Psi(x,t)$ (or "ket" $|\Psi\rangle$ in Dirac notation), describing any physically-realizable state of of a system and claiming that it contains all accessible physical information about that system [1] . Sometimes, this is accompanied by "The principle of Superposition" being added that for any physically realizable states, other states can be formed by linear superposition with complex valued coefficients. These complex coefficients stress the importance of the relative phases of the components being added [2]. The other postulates appear with different numbers from text to text so that their names (if provided at all) are more important than their numbers.

The ordering I will use here for the primary postulates of quantum mechanics are: again [P1] for the existence complex state functions, [P2] is about operators corresponding to observables, [P3] is the Schrödinger equation, [P4] says that measurement is a projection for resulting eigenvalues, [P5] is the Born Rule for outcome probabilities, and [P6] says that states of a composite system are tensor products of component states.

In my view, the strangest and most important postulate is the **"Born Rule" [P5]** implying a probabilistic interpretation for the wavefunction [3]. In one simple example, $|\Psi|^2$ is the probability that the system will have given coordinates at time t; and this in turn means that Ψ is a strange and new concept called a **"probability amplitude"** (at least in the usual Copenhagen interpretation of quantum mechanics). It also means that the sub-quantum world, if it indeed "exists" at all, lives in something like "the square root of reality." And that often makes it very different from any concept in classical physics. In

[1] Schrödinger initially intended his Ψ to correspond to a real wave, but Born's probability wave quickly prevailed instead. Students of Schrödinger wrote a poem: "Erwin with his psi can do, Calculations quite a few. But one thing has not been seen: Just what does ψ really mean? [Remembered by Felix Bloch].

[2] i.e., a complex coefficient can be written in polar form, $c_1 = a_1 e^{i\phi_1}$, where angle ϕ represents the relative phase difference of functions with respect to neighbors in the sum. They are all locked-in or entrained together with these fixed relative phases. An imaginary coefficient implies phase 90^o *or* $\pi/2$ radians, i.e., $i = e^{i\pi/2}$

[3] Max Born stated this conclusion as a footnote in a 1926 paper on particle collisions. Yes, I know that Schrödinger said that entanglement was the strangest and most distinguishing concept; but I would call it just a close second.

particular, it makes the use of complex and hypercomplex numbers seem to be a necessity (e.g., quaternions and Dirac matrices) [4]. It should be a goal of new research into the foundations of quantum mechanics to derive the Born Rule instead of simply postulating it.

One factor making the quantum postulates inconsistent from text to text is that they are a mishmash of main postulates (that are logically fundamental), secondary postulates (derivable from main postulates), and other "mere consequences" of postulates that happen to be well known (like the "uncertainty principle" which is just a derived principle rather than being fundamental) [1]. Inconsistent stress makes it unclear what is most important. Unlike the principles underlying relativity (see later section), the principles of quantum mechanics are exclusively in the language of abstract mathematics whose physical meanings are unclear. The corresponding physical principles are in dispute.

Complex numbers in quantum mechanics: Complex numbers appear almost everywhere in quantum mechanics and greatly facilitate calculations. Quantum formulations depend on the use of complex numbers in all textbooks. So, if you want to learn quantum mechanics, you have no present choice but to accept and use complex numbers. In disciplines such as electrical engineering, complex numbers are a great convenience in calculations; but final answers just use the real part. A strong majority opinion among physicists is that complex numbers are instead essential and intrinsic in quantum mechanics. In the discussions below, the initial choice of describing waves in complex polar form, $ce^{i\varphi}$, leads to the use of complex amplitudes and then complex operators. Adding waves of different shapes or frequencies means caring about the phase relationships between waves, and complex numbers do that well. The resulting mathematical system is highly dovetailed, self-consistent, and tremendously successful. There are still many dissenters who wish to structure quantum mathematics differently (such as using 2×2 matrices in place of complex numbers). But their attempted constructions generally increase computational difficulty and reduce economy of the mathematics. Ultimately, the dogma of complex numbers being intrinsic depends on finding a good interpretation of the quantum world (e.g., is the wave-function real in some sense (ontology) versus having it rather reflect "our knowledge" of a system (epistemology).

Here is an **Outline** of key points addressed in the sections that follow:

[4]For example, electron spin has a "Hilbert Space" of only two base vectors, $|up\rangle$ and $|down\rangle$ for a spin projection in say the "z" direction. But after a test with a Stern-Gerlach magnet, future spins can be measured in an x or y direction too. The two z-bases cover both of those cases (very un-real-vector-like behavior because spin is hyper-complex). If we let $a = 1/\sqrt{2} \simeq 0.707$, then x-spin right $= |\rightarrow\rangle = a|\uparrow\rangle + a|\downarrow\rangle$ and spin y or spin down into the paper $|\odot\rangle = a|\uparrow\rangle + ia|\downarrow\rangle -$ funny superpositions of up/down base states. And for y, a complex coefficient is really required. Also, the y-spin operator (Pauli Matrix) is complex: $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. This can be thought of as i times a quaternion, q_y.

- The simple plane wave is used to intuit differential **operators** on eigenfunctions to give eigenvalues. Conservation of energy is then written with operators to form the Schrödinger Equation of quantum wave-mechanics and the unitary evolution of state functions.
- Simple examples are given of superposition. The Born Rule explains chemical bonding by an extra enhancement of electron density between two nuclei.
- The uncertainty principle is derived as a consequence of primary postulates and shown in two forms.
- Further mention is made of the primary postulates of quantum mechanics: [P1] complex state function, [P2] corresponding operators, [P3] Schrödinger equation, [P4] projection for resulting eigenvalues, the Born Rule [P5], and tensor products for composite systems [P6].
- The elementary non-relativistic Lagrangian, $L = T - V$, is derived simply by **counting waves along a path** (making the simplest Feynman path integral easy to understand). And later, it is derived from the "principle of maximum proper time" along a path.
- Time dilation, $\Delta t = \gamma \Delta \tau$, is derived simply from the Lorentz metric designed to give zero interval when two events are connected by light.
- Length contraction is derived simply from the invariance of the metric (with a standard derivation shown in the appendix).
- The relativistic Lagrangian, $L = -m_o c^2/\gamma - V$, is derived simply by counting waves along a path.
- First order general relativity is explained simply by special relativity combined with the principle of equivalence.

Many of the items mentioned here and in the following are not well known and are generally hard to find in the literature. I had the joy of discovering them largely by myself. But I presume that they are familiar to "those who know well."

2. Background

I believe that the most important concepts in quantum mechanics should begin with stating:

$$[P0] \qquad \mathbf{p} = \mathbf{h}/\lambda = \hbar\mathbf{k} \qquad and \qquad \mathbf{E} = \mathbf{h}\nu = \hbar\omega \ ; \tag{1}$$

and these equations apply to both light waves and to matter waves [5]. I would call this Postulate Zero [P0], and it is a statement in the language of physics. No textbook incorporates this as a postulate largely because it suggests that waves have a reality; and that goes against the majority beliefs of the mid-1900's. But, if we wish for an intuitive understanding of QM, we must start with the belief that these waves are at least"real" in some sense, and that view is becoming increasingly more popular. A big question is "waves of what?" The orthodox answer is "waves of probability amplitude" − but that certainly

[5]Here, $\hbar = h/2\pi$, $k = 2\pi/\lambda$, $\omega = 2\pi\nu$ and non-relativistic momentum $p = mv$.

doesn't resonate in our intuitions. For light photons, the waves appear to just be vector electromagnetic or vector-potential waves. This is especially apparent when single photons refract through glass in the same way that classical electrical waves do [6].

The key equation $E = h\nu$ originated with Planck's 1900 paper on black body radiation for what were later called photons. The quantum idea was that if a frequency ν was present, then it was only capable of delivering a "quantum" of energy, $\Delta E = h\nu$ to an absorber. Then, in 1924, de Broglie used Einstein's relativity theory to claim that massive particles also obey this rule and that total energy determines a fundamental clock rate for electrons. He also claimed that an effective wavelength exists for moving particles, $\lambda = h/p$, extended from Einstein's idea that light quanta also possessed momentum [7]. We later said that massive particles with momentum and energy have associated de Broglie matter waves with a wavelength and frequency.

That electrons diffract from crystals just like x-rays was first shown by Davisson and Germer in 1927. It was then shown that matter wave diffraction also occurs for neutral atoms and now even large molecules like buckyballs (C_{60}). Instead of being physically "real" waves, the scalar "matter" wave might be understood as representing an information "code" where wave concentration in space can inform about momentum and wave peak density in time tells energy. A wave also has a phase-velocity given by $v_\phi = \lambda\nu = (2\pi\nu)(\lambda/2\pi) = \omega/k$. So, a wave moving to the right would be given by x-coordinate $x = v_\phi t$, *or* $kx = \omega t$. We pick a point or phase on the wave and follow its motion. So, if $y = a\cos\phi = a\cos(kx - \omega t)$ has say $\phi = 0$ (and peak $y = a$), we can follow its motion to the right.

What are the most important equations in mathematics? The Pythagorean theorem might be one answer (although it only applies when space is flat and not, say, on the curved surface of the Earth). But competing with that answer might be Euler's formula, $e^{i\pi} = -1$ (relating the number $e \sim 2.718$ and named "e" after Euler, and pi, and the 'irrational' number i). If it is that important, then almost everyone should know it. And it is a special case of $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, which gives Euler's formula when the phase is pi (180 degrees). Rather than having just real waves, we prefer to generalize to complex waves using the exponential with base e. This is not just for the usual calculus convenience; it is widely believed that complex numbers are intrinsic in quantum mechanics. So, for matter waves, we write: $y = a\, e^{i\phi(x,t)} = ae^{i(kx-\omega t)} = \mathbf{ae^{-i(\omega t - kx)}} = \mathbf{ae^{(-i/\hbar)(Et-px)}}$.

In quantum mechanics, we like to label our wave functions with amplitudes and phases by the symbol, "psi," Ψ. The equation above is for an infinitely long "plane" wave (over all x and all t). If it also represents the motion of a so-called "particle," we might want

[6] but we then have to ask ourselves which came first: classical EM waves down to the quantum level or intrinsically quantum EM ideas seeming classical due to large numbers of boson photons. That answer seems to be quantum first, and then build up from there.

[7] First noticed by Stark in 1909 as $p = h\nu/c$ and then finally and formally by Einstein in 1916.

to restrict its domain better and "localize" it. This usually involves some superposition of other wavenumbers, k (and discussed under the topic of Fourier analysis). A gaussian shape in space for the waveform associated with a particle would result from a bell-shaped profile of wave numbers, k. But, for the present, we will just look at the oversimplified plane wave.

Once we have wave phases in an exponent, we will wish to be able to pull down the values of E and p (their eigenvalues) from the expression for the wave. Obviously, this can be done by creating and applying "operators" $\hat{E}$ and $\hat{p}$ so that $\hat{E}\psi = E\psi$ and $\hat{p}\psi = p\psi$; we just make operators that work that way.

3. Simple Plane Waves:

Using a plane-wave traveling wave train as the most elementary heuristic example, we have a choice of expressing it as a wave in its own terms or in terms of energy and momentum as parameters a measurement observation might prefer.

$$\psi(x,t) = Ae^{-i(\omega t - kx)} \;\rightarrow\; \psi(x,t) = Ae^{-i(E_o t - p_o x)/\hbar} \tag{2}$$

Since $E = \hbar\omega = h\nu$, and $p = h/\lambda = \hbar k$, these equations are equivalent. In this equation, we have single constant values for the wavenumber, k, and the angular frequency, ω. It might be that the left equation happens to be the one preferred by Nature for a wavefunction in the spacetime between an emitter and detector and that the Planck constant, $\hbar$, might only enter when a (classical) detector 'collapses' the wavefunction to make use of its particle energy or momentum [8]. Perhaps the simple wave is everywhere a carrier of quantum information without physical actualization; and the density of wave peaks in space and in time represents information as a **'code'** about what might actually be detected as a physical particle.

Nature can also use this code to deduce a particle's rest mass, see for example equations (15) and (16) for m_o and ω_o later on. The 'particle' itself is only a deduction by the measuring apparatus and likely doesn't exist physically in the wavefunction. The amplitude of the wavefunction can disperse and weaken over time and distance and still carry the information ultimately used. Note that the units of h are $[h] = joules \cdot sec = J/hertz = [momentum]/wavenumber = [action]$. Each vibration per second contributes a unit of energy; each packed wavelength adds momentum.

Operators: If one begins with $\psi = \psi(x,t)$ as in equation (2), we then wish to retrieve the energy and momentum it contains in the exponential. Obviously, derivatives will pull these out. That is, creating an **operator** denoted as $\hat{\mathbf{p}} = -\mathbf{i}\hbar\,\partial/\partial\mathbf{x}$ (or $\hat{p} = -i\hbar\nabla$ in 3-dimensions) gives us a so-called "eigenvalue" equation $\hat{p}\psi = p_o\psi$. That is, the momentum operator on the wave function yields a constant times the wave function.

[8] It seems to me that should require some sort of sub-quantum network transaction or hand-shaking agreement between source and absorber. But due to a general avoidance of discussion about mechanisms, that is a minority opinion.

And using operator $\underline{\hat{\mathbf{E}} = (\mathbf{i}\hbar)\,\partial/\partial\mathbf{t}}$ gives us $\hat{E}\psi = E_o\psi$ [9]. This is a first example of another postulate in QM that might be called the "correspondence principle" [10] :

[P2] To every physical observable, there corresponds a linear operator (and we call it "Hermitian" or self-adjoint if it always results in a real value for the observable [11]). It is said that a difficulty in understanding QM is that instead of momentum being a deterministic variable, it now IS an operator operating on a wave function. I think what that means is that the operation of the operator interprets and enables activation of the underlying code contained in the wave function (the density of wave peaks over distance).

So far, there is nothing proprietary about having linear operators for quantum mechanics. We can also have them for classical waves too. In equation (2), for example, we could create operators that pull down the value of the angular frequency, omega, or the wave-number, k, by using $\hat{\omega} = i\,\partial/\partial t$ and $\hat{k} = -i\,\partial/\partial x$. Then, $\hat{\omega}\psi = \omega\psi$ and $\hat{k}\psi = k\psi$. Again, this is an example of a postulate for what is called a "linear eigenvalue equation" associated with each linear operator [2]. Ψ is called an eigenfunction of the operator, and the real constant is called the eigenvalue. It is also a postulate that [P4] "one or another of the eigenvalues is the only possible result of a precise measurement of the dynamical variable represented by" the linear operator [2].

Some texts consider **Schrödinger's equation** of 1926 ("SE," eqn.(3) below) as a primary postulate of QM, [P3]. Here, it is simply obtained by writing out conservation of total energy for a single particle in terms of these new operators, $\hat{p}$ and $\hat{E}$, on a wave function. Since kinetic energy $KE = mv^2/2 = p^2/2m$, the operator for KE will be $\hat{KE} = \frac{1}{2m}\hat{p}^2$; and potential energy $\hat{V} = V$. And these operate on the wave function, $\psi(x,t)$:

$$(3)\;\; KE + V = \frac{p^2}{2m} + V = E_{total} \;\rightarrow\; -\frac{\hbar^2}{2m}\nabla^2\psi(x,t) + V(x,t)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \quad [P3].$$

It has always seemed to me that this simple approach is the best way to intuitively introduce the Schrödinger equation for the first time rather than just postulate the strange complex-looking Schrödinger equation and have it sprung onto a first time reader [12]. What the $\hat{p}$ operator does is look at the density of wave peaks in space, and the $\hat{E}$ operator looks

[9] And for angular momentum, L, one considers change of phase around a circular phi direction, $\partial/\partial\phi$, or in more generally in 3D by $-i\hbar\, r \times \nabla$.

[10] But, the term "correspondence principle," is also used to state that the predictions of QM reduce to those of classical mechanics in the limit where a system approaches large quantum numbers or higher energies.

[11] Unlike the complex quantum world, the classical world only desires real results.

[12] And Weinberg's text on QM does touch on this heuristic introduction [3]. His book is also one of the few to mention interpretations (section 3.7) − but only for Copenhagen, Many Worlds, older hidden variables, and Decoherent Histories. He adds: "My own conclusion (not universally shared) is that today there is no interpretation of quantum mechanics that does not have serious flaws, and that we ought to take seriously the possibility of finding some more satisfactory other theory, to which quantum mechanics is merely a good approximation."

at the density of peaks in time. I've always thought that Nature must also do this by phase comparisons over small space-time regions. So, from a wave, Nature can deduce E and p.

The observable operator interpreted to mean energy (such as $KE+V$), is a distinguished observable called the 'Hamiltonian,' $\hat{H}$. So the Schrödinger equation can also be written as:

$$\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t), \quad or \quad \frac{\partial\psi}{\partial t} = \frac{-i}{\hbar}\hat{H}\psi \quad so, \quad \psi = \psi_o e^{-iHt/\hbar} = U(t)\psi_o(x,0) \tag{4}$$

where U(t) is a unitary time evolution operator that can be written as an exponential. So, we can use the Hamiltonian to give the time evolution of the wavefunction, $\psi(x,t)$. In some texts, this **unitary evolution with time, U(t)**, is given as a primary postulate **also [P3′]**, and the SE follows from it. Hamiltonian energy isn't always $\overline{KE+V}$, there are other forms too. For example, a particle with a magnetic moment in a magnetic field may have the form: $\hat{H} = \mu B \cdot \sigma$ (where σ refers to Pauli matrices). And when electromagnetic fields are present, $\vec{p}$ becomes $\vec{p} - e\vec{A}$. But, for any Hamiltonian energy, $\psi(x,t)$ evolves continuously and deterministically into the future, until the point where final measurement occurs. Then the wave function collapses, and determinism is lost. In other words, "a great miracle occurs," and nobody really knows how.

Unlike the simple example above, in traditional classes the Schrödinger equation is simply presented as (an initially strange) founding postulate of non-relativistic quantum mechanics. Its solutions include tunneling, complex atoms, and s-orbitals which no longer resemble anything like plane waves. For example, just try a solution resembling an exponentially decaying radial profile: $\psi_1 = Ae^{-b\,r}$ and plug that into the SE with an atomic nucleus central potential $V = -Ze^2/4\pi\epsilon_o r$ and use $\nabla^2\psi = r^{-2}\partial/\partial r(r^2\partial\psi/\partial r)$. And then solve for the actual coefficients A and b. The result is the normalized [13] 1S atomic orbital:

$$\psi_1(r) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_o}\right)^{3/2} e^{-Zr/a_o}, \tag{5}$$

where $a_o = 4\pi\epsilon_o\hbar^2/me^2$ is the first Bohr orbit $\simeq 0.53\mathring{A}$, and Z is the proton number [14]. And then there is also multiplication by a time varying factor with a frequency given by $\nu = E/h$. $\psi_1(r,t)$ is like a tent shape that is up and then becomes inverted down and then back to up again − but in 3D. This profile is like nothing experienced in the classical world, and there is nothing orbiting in the orbital. So, even though the Schrödinger equation makes simple intuitive sense for plane waves, its application goes well beyond that.

[13] In quantum mechanics, normalizing means finding a front end coefficient such that the integral of $\psi^*\psi$ over all space gives 1 = 100% total probability.

[14] This is the innermost atomic orbital that Bohr missed in his early theory where electrons were standing waves about a nucleus. Actually, he only quantized the orbital angular momentum without yet realizing that it could represent de Broglie waves. And he started with angular momentum 1, 2, and up; while the 1s state has orbital angular momentum $L = 0$.

So, introducing it as a postulate makes some sense [15].

This particular atomic 1S orbital can be superimposed with other orbital functions and still be an appropriate combined wave function. An example is the single electron shared by two hydrogen nuclei, A and B for the simplest molecule $H_2{}^+$, with binding wave function $\Psi_+ = a(1S_A + 1S_B)$. The "binding" itself is due to the overlap of these two spherical functions followed by squaring (the special enhancement of electron density in-between the two protons due to application of the Born Rule). Another is carbon's four-valence-electron hybrid orbital [Linus Pauling, 1931]: $\Psi_1 = 0.5(2s + 2p_x + 2p_y + 2p_z)$ where the p-orbitals have angular momentum $L = 1\hbar$. In both cases, the coefficient signs (+ in this case) are very important because they represent the coordinated phases of the superpositions. The result for carbon is the formation of a lobe of enhanced electron density sticking out from the carbon atom in the direction $\hat{i} + \hat{j} + \hat{k}$. For all four electrons, we get four tetrahedral spaced lobes ready for bonding (e.g., like for methane, CH_4). In carbon, rather than have all those individual orbitals vibrate separately, it makes sense for them to get entrained together (entangled or hybridized) so that they have more aspects of constructive interferences.

Some say that the uncertainty Principle is a key property of QM, and that is occasionally introduced as a postulate too. But it is actually just a derivation from other more key postulates. In the oversimplified case of a plane wave, there is no localization of any presumed particle. Localization can be expressed with a wave-packet which can be created from a Fourier distribution of plane waves. If the wave-packet has a spatial width (say the standard deviation for a Gaussian packet), then the uncertainty principle applies in either form for the widths of x versus p or for x versus wavenumber, k (i.e., quantum mechanics not required).

That is, somehow, Nature effectively can perform the equivalent of Fourier Transforms (going from waveform in space or time to wavenumber or frequency in space or time). It is not clear how it does this, but it explains the Heisenberg Uncertainty Principle. That is, let wave-packet shape have an associated Gaussian probability envelope such that its probability density, $P_x = dP/dx$, is described by:

$$P_x \propto e^{-x^2/2{\sigma_x}^2},\ so,\ \psi(x) = \sqrt{P_x} \propto e^{-x^2/4{\sigma_x}^2} \tag{6}$$

The symbol sigma refers to "standard deviation" or square-root of variance in statistics. The Fourier Transform (FT) of a Gaussian is itself a Gaussian so that the momentum wavefunction $\phi(k) = \sqrt{P_k} \propto \exp(-k^2/4{\sigma_k}^2)$. Since $\exp(-a^2x^2) \longleftrightarrow \exp(-k^2/4a^2)$ is a transform-pair where $a^2 = 1/4{\sigma_x}^2$, we have:

[15] There is a Fourier Transform from the 1S exponential decay wave, but it is a 3-D spherically radial transform not easily associated with plane waves. The form in momentum space is $\phi(\vec{p}) \propto p/(p^2 + 1)$ [4]. One has to integrate $\int \psi(r) exp(-ip \cdot r) dr$.

$$\frac{k^2}{4(1/4{\sigma_x}^2)} = k^2{\sigma_x}^2 = \frac{k^2}{4{\sigma_k}^2},\ \Rightarrow\ \sigma_x\sigma_k = \frac{1}{2},\ \Rightarrow\ \sigma_x\sigma_p = \frac{\hbar}{2},\ \ or,\ \Delta x\Delta p = \frac{\hbar}{2}. \tag{7}$$

The case of Gaussian envelopes is optimal and gives equality. Any other waveform envelope profile will give $\Delta x\Delta p > \hbar/2$. A distribution of momenta in a wave packet will cause spreading of the spatial width of the wave packet over time

In the case of just waves without momentum being considered, Fourier transform theory says, $\Delta x\Delta k > 1/2$. That is, an uncertainty principle applies to waves by themselves without any mention of Planck's constant, $\hbar$. In electrical engineering, "It is well known that the bandwidth-duration product of a signal cannot be less than a certain minimum value" [5]. That is, $\Delta t\,\Delta freq \geq 1/4\pi$ or $\Delta t\,\Delta\omega \geq 1/2$. So, if a special class of electrical engineers had existed in 1927, there wouldn't have been so much mystery about these uncertainty principles.

Another important concept in quantum mechanics is the use of the "commutator bracket" of two linear operators: $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$. Most of the time in classical mechanics, the commutator will be zero. But, in QM, $[\hat{x}, \hat{p}_x]\psi = -i\hbar(x\partial_x\psi - \partial_x(x\psi)\ ) = i\hbar\psi$. This is used in a general form for uncertainty relations, $\Delta A\ \Delta B \geq \langle[A, B]\rangle/2i$ so that $\Delta x\Delta p \geq [\hat{x}, \hat{p}]/2i = i\hbar/2i = \hbar/2$. But again, neither "commutators" nor "the uncertainty principle" are unique to quantum mechanics, they also appear in usual classical physics [6]. For example, for classical waves, $[t, \partial_t] = -\mathcal{I}$. What is unique to QM is the appearance of the value $\hbar$, the concept of "entanglement," the existence of probability amplitudes, and the phenomenon of "collapse" of the wave-function and the apparent reification of particle behavior.

4. The Postulates

The first postulate of quantum mechanics is sometimes stated more elaborately as:
[$P1$] For every system, there is a corresponding **Hilbert space**, $\mathcal{H}$ [16]; and a state of the system is a unit ray in the Hilbert space.
A student has to understand this statement but might also ask why it is written in this

[16] In 1932, von Neumann decided to include a collection of states into a "Hilbert Space" from a publication in 1924 by Courant and Hilbert (for pure mathematics purposes). A Hilbert space is an abstract vector space having an inner product. The simplest example is the ordinary real Euclidean vectors with unit vector basis $\{i,\ j,\ k\}$ and the familiar dot product $\vec{A}\cdot\vec{B} = |A|\,|B|\cos\theta$. In quantum mechanics, we can add that it is a "function vector space" [such as spherical harmonics, Hermite polynomials, or Legendre polynomials (1782) on the interval $-1 \leq x \leq 1$] and referred to under the heading of "Lebesgue spaces, L^2 " of square integrable functions. But quantum mechanics allows for complex coefficients. This applied mainly to Schrödinger's Wave Mechanics. Heisenberg's Matrix Mechanics came a little earlier in 1926 and was formulated with potentially infinite square matrices with a Hilbert Space of sequences of complex numbers: "little ℓ_2" spaces. The two formulations are mathematically equivalent. Note that Hilbert space can be, and often is, composed of an infinite number of bases. So, a vector can be a sum of an infinite number of components.

initially opaque language. Hilbert space includes linear vector spaces, so the usual mathematics from a course in "Linear Algebra" is automatically implied. This, of course, includes the superposition principle (adding vectors together still gives a vector). The "vectors" in this case are more commonly functions that can be added and subtracted in the same way as vectors (except that phase is also important).

Two State Superpositions: One of the simplest examples of postulate [P1] is the polarization states of the photon (e.g., see Feynman Lectures Vol. III [15]). For the case of a photon traveling in the z-direction, the Hilbert Space of this single photon system can simply consist of two basis vectors called $|x\rangle$ and $|y\rangle$. That means that the electric field vector is perpendicular to the direction of motion and can be in the "horizontal" x or "vertical" y directions. Any other direction (like 45^o) is a real superposition of these base states. A horizontal polarizer will not pass the $|y\rangle$ state. But, we also like to say that a photon carries spin; and this can be written as right or left circularly polarized states by a complex superposition of the base states:

$$|RHC\rangle = |R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle), \qquad |LHC\rangle = |L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle), \tag{8}$$

where the $1/\sqrt{2}$ coefficients "normalize" the states (i.e., $R^*R = 100\%$ and $L^*L = 1$). The imaginary coefficient i says that the addition of the y sine wave is 90^o out of phase with the x sine wave (so the y-wave is cosine). When that happens, the tip of the electric field vector, $\vec{E}$, rotates about in a helix and carries angular momentum.

Now, equation (8) can be inverted to solve for x and y in terms of R and L. That means that R and L could also be considered as the bases for the Hilbert space. So, which is more "real?" Both selections are equally valid with utility varying with the nature of the experiment observing the photons (e.g., polarizers or quarter-wave-plate/Nichol-prism combination, etc.). Single photons can be either circularly (or elliptically) polarized or linearly polarized. So, how can photons have spin $S = \pm 1\hbar$ and also be linearly polarized? The answer is that a linearly polarized photon (spin zero) can be considered as a superposition of both forward and reverse spin (RHC and LHC) at the same time. That is OK in quantum mechanics.

And, as if that wasn't counter-intuitive enough, we can also have macroscopic cases of persistent currents in superconducting ring loops that can exist in a superposition of both clockwise and anti-clockwise directions of current flow at the same time.

"State" is a key word in quantum mechanics. In Schrödinger "wave mechanics" it may also be called a "wave function." A traveling (time dependent) state is a mathematical expression for a matter-wave that represents an appropriate relation or transition between a source and a detector and possibly what's in-between. It has to go through mathematical processing's before it can be said to have any classically understood "reality" (unitary evolution, "reduction," Born rule "squaring,"...). There are also time-independent states such as the hydrogen atom orbitals, and these can be considered as "standing-waves." Exactly what a state means and how "real" it is in itself has been a source of continuing discussion

and ongoing contention. The mathematics has always worked perfectly, but what a state represents to us is somewhat opaque. In a sense, it tells all of the possible outcomes from a measurement. Examples include the interference output from slits in two-slit diffraction, a moving electron, electron spin, atoms, and molecules. Measured values of an experiment are called eigenvalues which are intrinsically classical and real and are not properties of quantum objects which are complex [7].

The most desirable background for studying quantum mechanics is a mathematical knowledge of linear algebra. In older days, students simply picked this up during the learning of quantum mechanics. This has the advantage that only a portion of linear algebra is needed, and in physics that portion is in the desired notation and application (our vectors are in Hilbert space, a term barely mentioned in math books). One nice source for learning this is in Griffiths [18]. After a course of study, one should be able to easily say things like, "two unequal eigenvalues of a Hermitian operator have orthogonal corresponding eigenvectors." In most cases of interest, a state function $|\psi\rangle$ will have a variety of eigenvalues and eigenfunctions − not just one as for the simple plane wave. Then, a better statement of the Born Rule [P5] is: given that a system is prepared in a state $|\psi\rangle$, the probability of seeing a measured system in an eigenstate $|a\rangle$ for an observable $\hat{A}$ is given by $P_a = |\langle a|\psi\rangle|^2$. We have to know about inner products, linear functionals and the dual space, operators, projectors, subspaces, orthonormal bases, matrices, diagonalization of Hermitian operators, traces, density matrices, probability theories, tensor products, and much more [18]

A very key (and very confusing) term in quantum mechanics is "measurement." One view is that it is a projection operating on the wave function and always causing the system to jump into being an eigenstate of whatever dynamical variable is being measured. The measured result is the eigenvalue of that eigenstate. A measurement actualizes values for the state. A standard view is that a state, ψ, **collapses** its wave-function in the act of measurement. A wave-function may be spread over kilometers (or possibly even light-years) but then has to suddenly everywhere collapse into a point for measurement detection: "The electron or photon ended up Here!" Possible mechanisms for doing this are presently unknown, and there are many conceptual difficulties (the "Measurement Problem"). If we are expecting an explanation to connect the measurement outcome to some property of a particle before the time of measurement, the problem might be in the word "before" (presumptions about the nature of quantum information and time). And the term collapse might be replaced by other suitable conditional probabilities in consistent theories [18].

Another way of stating measurements [P4] is the "von Neumann Postulate: If a measurement of the observable A yields some value a_i, the wave function of the system just after measurement is the corresponding eigenstate ψ_i [1]. This is another non-deterministic discontinuous collapse due to the act of observation by projection of a superposition to one

of its terms.

In addition to the "main postulates" already mentioned ([P1] complex state function, we again also have [P2] corresponding operators, [P3] wave equation, [P4] projection for resulting eigenvalues, and the Born Rule [P5]) , one can derive "secondary postulates" [1]. These include superpositions, eigenfunction and eigenvalues, calculation of expectation values, expansion in eigenfunctions, and conservation of probability.

A further note on postulate [P6] Tensor Products: A primary postulate that sometimes goes unmentioned as primary is about "tensor products": the state of a composite system is in the direct product of the Hilbert spaces of its component systems: $S = S_A + S_B \Longrightarrow \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. This is important when discussing entanglements for two or more particles. For example, the state $|\Psi\rangle = (1/\sqrt{2})(\ |1\rangle_A \otimes |0\rangle_B - |0\rangle_A \otimes |1\rangle_B\ )$ is an entangled state.

Historically, Heisenberg's "Matrix Mechanics" came slightly earlier than Schrödinger's "Wave Mechanics," (1925 and 1926). But Schrödinger's mathematics was much easier to use and quickly gained popularity. Physicists were familiar with the language of differential equations, but few knew anything about matrices [17]. They were difficult to apply for most common problems and now find use in fewer applications (such as the harmonic oscillator). In 1926 to 1930, Dirac invented his more general "transformation theory" and could derive both the wave and matrix pictures from it (and operators now become the generators of transformations). Then in 1939, Dirac introduced his now common notation of "bra" and "ket" vectors, $\langle\varphi|$ and $|\psi\rangle$, with "inner product" then conveniently written as $\langle\varphi|\psi\rangle$. This is the analog of the usual "dot" product of vectors, $\vec{A}\cdot\vec{B} = |A||B|cos\theta$. But for continuous functions, it might look more like $\int \varphi^*\psi\ d(volume)$.

In addition to the wave and matrix formulations, there is also a "path integral" or "sum over histories" formulation from Richard Feynman (sum over all possible paths that a particle could take weighted by phases along each path). Feynman wrote a technical book on this [13], but he also discussed an elementary version in his much more popular book called "QED" [14]. From his formulation, he derived the Schrödinger equation. The relevant phases depend on "action," $A = \int Ldt$, where L was a slightly opaque function called a "Lagrangian."

Path Integrals and Least Action: Derive the Lagrangian $\mathbf{L} = \mathbf{T} - \mathbf{V}$ simply by counting waves along a path (making the simplest Feynman path integral easy to understand):

This is based on the "principle of least action" or "principle of stationary action" which dates at least back to 1662 for "Fermat's Principle" for light rays and to 1744 for massive

[17]Essentially, they had never been used by physicists since their discovery by Cayley in 1855 and were considered as "pure mathematics."

particles [Maupertuis and Euler]. The equations they used were:

$$\delta \int_{t_1}^{t_2} 2T(t)dt = 0 \qquad and \qquad \delta \int_{x_1}^{x_2} pdq = \delta A = 0. \tag{9}$$

where T is another symbol for kinetic energy, KE, "q" is a generalized coordinate symbol for distance (most of the time we could just use " x" instead), A is "action" (the integration over time), and trial paths are varied (symbolized by "change in" or delta . The game we play is to fix end points at time-1 and time-2 and then vary paths in-between until they satisfy requirements (called "calculus of variations" in mathematics with solutions given by the "Euler-Lagrange" equations). These concepts were then broadened by Lagrange [1760] and Hamilton [1835] where the integrand came to be called the "Lagrangian", $L = L(x, \dot{x}, t)$ which is often just $L = T - V$ (and here $\dot{x} = dx/dt-$ Newton's notation for time derivative). Anyway, the purpose of all this was to have an alternate but mysterious formulation of Newtonian Mechanics [18].

The two forms above in equation (9) are inter-related: that is, Euler's action was the integral of pdq or

$$pdx = mv\, dx = m\frac{dx}{dt}dx = m\frac{dx}{dt}dx\frac{dt}{dt} = m\left(\frac{dx}{dt}\right)^2 dt = mv^2dt = 2T\, dt. \tag{10}$$

It matches, but If we wish to "extremize" paths, a differing constant of proportionality wouldn't matter.

It wasn't perfectly clear why this approach worked or what it might really mean until it was applied to quantum mechanics and waves. The action then becomes proportional to the total number of waves along a path (or total phase), and the best path is one that provides the most constructive wave interference at the end points. So now, lets just forget some of this previous history, and work backwards to find an action and Lagrangian that allows this to happen for a single free particle.

Since $p = h/\lambda$, a wave-count along a path is $n = \Delta x/\lambda = p\Delta x/h = 2T\Delta t\ /h$, just like the transformation of pdx in the above equation (10). Nearby paths with nearly equal wave phases or counts, n, will have good constructive interference and be preferable and stationary.

To complete the counts calculation, note that total wave phase is seen in equation (2) as $\phi = (kx - \omega t) = (px - Et)/\hbar$, where E = T+V [19]. So, wave counts is:

$$n = \frac{\Delta\phi}{2\pi} = \frac{(p\Delta x - E\Delta t)/\hbar}{2\pi} = \left(\frac{2T - (T+V)}{h}\right)\Delta t = \frac{T-V}{h}\,\Delta t = \frac{L}{h}\Delta t. \tag{11}$$

[18] For more discussion on Least Action, see the Feynman Lectures on Physics [15].

[19] Remember, we are dealing with non-relativistic mechanics, so mass energy is not included. If it were, then intrinsic vibrational frequencies would be huge– almost beyond measurement.

Where the elementary classical Lagrangian is $\mathbf{L} = \mathbf{T} - \mathbf{V}$.

5. Relativity

The fundamental ideas of special relativity (SR) can now be found in any of hundreds of basic books on elementary modern physics, and those approaches will not be stressed here. Historically, the ideas for length contraction and time dilation go back at least to the works of Lorentz, FitzGerald, and Poincare on the properties of Maxwell's equations, the ether and the null result of the Michelson-Morley experiment (1887). "Lorentz Transformations" were in common use prior to Einstein, but their interpretation wasn't clear. Einstein's 1905 paper on the electrodynamics of moving bodies changed beliefs by using two central simplifying assumptions:

"The Principle of Relativity:" physical laws are invariant with respect to frames of reference in uniform motion relative to each other, and,

"The Principle of Invariant Speed of Light:" Light speed $\mathbf{V} = \mathbf{c}$ regardless of the uniform motion of either an emitter or observer.

This different point of view made the previously all-important "luminiferous aether" now seem superfluous [20]. Note that these postulates are "in the language of physics" rather than abstract mathematics. It is hoped that quantum mechanics may someday be derived from similar physical principles. Rather than the term "relativity," Einstein later wished that he had used the term "invariance" instead (e.g., the laws of physics should be invariant under Lorentz transformations – a symmetry principle). It is also implicitly understood that the space we live in is isotropic and homogeneous (and this gets carried over to cosmology as well).

Here, I would like to approach the subject of relativity and time dilation in a slightly different way beginning with the concept of metric. In geometry, we can look at distances as the positive value of the separation of two marks on a measuring tape. In relativity, we shift from "marks" in space to **"events"** which take place in both space and time (four coordinates or 4-dimensions). And we compare the separation of two events in terms of a transit of a **beam of light** between events. This is a profound difference in views. The standard mathematics for "metric spaces" insists that distance measures be positive. In relativity, we break this rule and consider both positive and negative distances and treat the signature of space differently from that of time.

[20] But Einstein changed his mind about the aether after the success of General Relativity of 1915. For him, aether was now the geometry of space-time, $g_{\mu\nu}$, later on to include other fields as well.

The important Pythagorean theorem for right triangles on a plane states $a^2 + b^2 = c^2$; for example, sides of length 3 and 4 give a hypotenuse of length 5, i.e., $3^3 + 4^2 = 5^2$. A distance between two points of 5 units will be preserved regardless of the coordinates being used. A student would be incredulous if someone claimed that the interval between these same two points instead obeyed a metric looking like $distance^2 = 4^2 - 3^2$, but something like that happens in special relativity. The usual Pythagorean idea can be extended to three-dimensions: $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = \ell^2$ as a Euclidean (E^3) metric distance. For more general cases, we use small increment change, "$d = tiny\ \Delta$" distances, and allow arbitrary coordinate systems (e.g., cylindrical or spherical) and summarize weighted coefficients symbolically.

In conventional notation, we now write: $dx^2 + dy^2 + dz^2 = \mathbf{ds^2} = \mathbf{g_{ij} dx^i dx^j}$ with orthogonal coordinates now labeled dx^1, dx^2, dx^3 and superscripts $i\ and\ \mathrm{J}$ just standing for coordinate values 1, 2 and 3 (which could mean x, y, z). The new coefficients, g, refer to "metric tensor." So, for usual E^3 space, the metric coefficients are just trivially $g_{11} = 1$, $g_{22} = 1$ and $g_{33} = 1$, with all other indices $g_{i \neq j} = 0$. We call this a "diagonal" metric." The term $\mathbf{ds^2}$ is called an "interval" and is invariant. We can select any coordinate system in E^3 to specify the coordinates of any two given points. We can rotate and translate the orthogonal axes in any way; and the resulting distance interval Δs^2 will be the same. This concept carries over to the invariance of relativistic space-time intervals; and this can be used to simplify calculations (as shown below for length contraction).

In general relativity (GR), the idea of gravity is replaced by curvatures deriving from a general 4-dimensional space-time metric called "$g_{\mu\nu}$" shown like a 4×4 matrix of values and called a metric tensor (with "g" for gravity). We let the subscripts $\mu\ and\ \nu$ stand for values 0,1,2 or 3 where the index "0" is reserved for time and 1,2,3 for space coordinates. The g's can be functions rather than constant values and can represent curvatures of space-time. The tricky thing about both SR and GR is that the metric distance doesn't have to be positive and its components for space and time can have opposite signs! This is often hard to grasp and goes under the name "pseudo-Riemannian-metric". So, what's that all about?

The big change for special relativity is that instead of usual distances, we now care about and focus on **"light" with speed c as a fundamental reference**. If two events in space-time are connected by a beam of light (or other massless radiation), we now want their separation interval to be called **"zero!"** For the general case, the metric this time is the **difference** between time and space increments: e.g., $ds^2 = c^2 dt^2 - dx^2$. A 'time-like' convention uses a plus sign on time (sign $g_{oo} = +1$) and minus sign on space, and $ds^2 = c^2 d\tau^2$ where τ is called 'proper time' meaning time in the frame of a moving object. For light, $dx = cdt$, $so\ ds^2 = 0$ as desired. So, the change from classically traditional positive metrics to difference metrics is due to the change of reference to light.

Time Dilation: A particle having mass will move more slowly than the speed of light, $v < c$, and we can write our metric as

$$ds^2 = c^2\, d\tau^2 = (g_{oo} = +1)c^2\, dt^2 + (g_{11} = -1)(dx^1)^2 \text{ or:}$$

$$(12)\quad d\tau^2 = dt^2 - \frac{dx^2}{c^2} = dt^2\left(1 - \frac{(dx/dt)^2}{c^2}\right) = dt^2\left(1 - \frac{v^2}{c^2}\right),\ so\, dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} \equiv \gamma d\tau.$$

This is "time dilation," and it can be picked off straight from the Lorentz SR metric form. The 'Lorentz factor' $\gamma \geq 1$; so, for example, if $v = 0.95c$, then $\gamma = 3.2$. Perceived time duration is larger than the clock time in the frame of the moving object. Then, for example, $\Delta t = \gamma \Delta\tau$ means that a muon with short half-life streaking through our atmosphere can live longer than it would at rest and be able to make it all the way through our atmosphere to the ground.

Also note that since light travels at speed v = c, the Lorentz factor is $\gamma = \infty$ so that $d\tau = dt/\infty = 0$. So, even though two events may be light-years apart, in the frame of a photon there is no advancement in time. Time flow is Zero (and, as intended, the "interval" $ds^2 = 0$). A photon leaves its source, "snaps its fingers," and instantaneously arrives at its absorber.

Length Contraction: Perhaps the simplest example of SR length contraction in the direction of motion is based on general interval invariance. Imagine a longitudinal bar in system S' of length L' moving to the right with velocity v relative to system S. Let two small flashes (events) occur when the leading and then the trailing edges of the bar coincide with a fixed post in S. $ds^2 = ds'^2 = c^2\Delta t^2 - \Delta x^2 = c^2\Delta t'^2 - \Delta x'^2$. Since $\Delta x = 0$ in S, the Δt is proper time $= \Delta\tau = L/v$. $L' = v\Delta t'$ and $\Delta x' = L'$. Then:

$$(13)\qquad (cL/v)^2 - 0 = (cL'/v)^2 - L'^2, \quad L^2 = L'^2(1 - v^2/c^2), \quad L = L'/\gamma.$$

Again, this is consistent with time dilation:

$$(14)\qquad \Delta\tau = \frac{L}{v} = \frac{L'}{\gamma v} = \frac{v\Delta t'}{v\gamma}, \quad \Delta t' = \gamma\Delta\tau.$$

The rest frame, S, sees a moving bar contracted along its length by the factor γ.

We now have the two key equations of special relativity, time dilation $\Delta t = \gamma\Delta\tau$ and length contraction, $L = L'/\gamma$. From these, the standard formulas for the "Lorentz Transformation" can be derived; and this can be used to show the invariance of the speed of light. Textbooks usually do this in reverse: use Einstein's postulates to derive Lorentz Transformations and then show time dilation and length contraction and then velocity transformations and relativistic kinematics. One can read textbooks for all of that.

The most famous formula in physics is $E = mc^2$, an idea dating back at least to Poincare in 1900 − but then only for electromagnetic fields. Einstein is generally given credit for this formula from 1905, but this and many other later publications by him either had important mistakes [21] or were incomplete [17] (he was generally sloppy about mathematics). The first complete proof of $E = mc^2 = \gamma m_o c^2$ was provided by Max von Laue in 1911. Correct kinematic derivations are now known by every college freshman in physics and will not be shown here.

Rest Mass: Energy is about the most important concept in physics. The rest mass of a particle is a fundamental vibration, $\hbar\omega_o = E_o = m_o c^2$. In special relativity (SR), we start with total (rest + kinetic) energy $E = mc^2 = \gamma m_o c^2$ and momentum $p = \gamma m_o v$, then:

(15)
$$E^2 = (\gamma m_o c^2)^2 = \frac{(m_o c^2)^2}{1 - \frac{v^2}{c^2}} = (m_o c^2)^2 \left[1 + \frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}\right] = (m_o c^2)^2 + (\gamma m_o v c)^2 = (m_o c^2)^2 + (pc)^2.$$

The same process can be repeated for frequency, $\nu = \gamma\nu_o$, and we differentiate between group velocity, $v = v_g$, and phase velocity, $v_\phi = \lambda\nu$, and the product $v_g v_\phi = c^2$. Then, we get:

$$\nu^2 = (\gamma\nu_o)^2 = {\nu_o}^2 + \nu^2 c^2 \frac{v^2}{c^4} = {\nu_o}^2 + \left(\frac{c}{\lambda}\right)^2, \; or \;\; \boldsymbol{\omega^2 = {\omega_o}^2 + (kc)^2}. \tag{16}$$

This can be conveniently pictured by right triangles having hypotenuse E with sides $(m_o c^2)$ and (pc) (or hypotenuse ν with sides ν_o and (c/λ)) [22].

Either way, if E and p are known, then m_o rest mass is also known from the wave code. And if frequency ν and wavelength λ are known, then rest frequency ν_o is also known. If $\omega/k = d\omega/dk = c$, then $m_o = 0$. So waves carry all this information even with very low amplitude. Redundantly, the knowledge of rest masses for the elementary particles is built into and accessible from the quantum fields of the Vacuum.

So, a wave enables determination of momentum or energy despite having weak amplitude, uncertainty is built into any wave-packet, and also rest frequency (or rest mass) can be deduced by the form of the dispersion relation, $\omega = \omega(k)$, which now also includes $E = E(p)$.

[21] The fact that Einstein's proof was not correct is detailed in the paper "Derivation of the Mass-Energy Relation" by Herbert E. Ives, Journal of the Optical Society of America v.42, p. 540 (1952).

[22] This is equivalent to the "on mass shell" 4-vector form $p_\mu p^\mu = (m_o c)^2$, or $c^2 p_\mu p^\mu = E^2 - (pc)^2 = {E_o}^2$ (also called the "mass hyperboloid" equation). Real observable particles have momentum vectors on-shell; but so-called virtual (internal Feynman line) particles have off-shell momenta.

The Schrödinger equation is non-relativistic with $KE = p^2/2m$. In that case, angular frequency would be written as $\omega = \omega(k) = (\hbar k)^2/2m + V(x)/\hbar$. Then group velocity is $v_g = v = \partial\omega/\partial k = \hbar k/m$, and phase velocity $v_\phi = \omega/k = \hbar k/2m + V/\hbar k$. Then $v_\phi = v/2 + V/\hbar k$. If the potential was not included ($V = 0$), then $v_\phi = v/2$ would seem very non-physical) and $\nu\lambda = v/2$ for the free particle. $p = h/\lambda = mv = mv_g$, so mass $m = h/v_g\lambda$. Of course, the non-relativistic case ignores the intrinsic frequency of rest mass.

Special Relativistic Lagrangian by Counting Waves: $L = -m_o c^2/\gamma - V$ appears implausibly different from the previous $L = T - V$ form. But, viewed from wave counts or total phase along a path, it becomes simple and almost obvious. We reuse the previous conversion from Euler's integral of $p\,dx \to mv^2dt$ from equation (10). Relativistic energy is now $E = \gamma m_o c^2 + V$ (from the equations under the section on "rest mass"); and $p = \gamma m_o v$ so that $pdx \to \gamma m_o v^2 dt$. Then, wave counts becomes:

$$n = \frac{\Delta\phi}{2\pi} = \frac{(p\Delta x - E\Delta t)/\hbar}{2\pi} = \frac{-\gamma m_o c^2 \Delta t(1 - v^2/c^2) - V\Delta t}{h} = \frac{-m_o c^2 \Delta t}{\gamma\ h} - \frac{V\Delta t}{h}. \tag{17}$$

And Least Action can be written as $\delta A = \delta \int L dt = 0$, where

$$A = -m_o c^2 \int_{t_1}^{t_2} \frac{dt}{\gamma} - \int_{t_1}^{t_2} V dt. \tag{18}$$

There are many more important topics in special relativity that could be discussed here. But, now we wish to move ahead to the topic of general relativity [GRT] and see if it can be easily and intuitively approximated.

The 'Principle of General Relativity' or "Principle of Equivalence" [PE] says that a local inertial system experiencing a constant gravitational force is equivalent to a noninertial system undergoing constant acceleration (relative to the"fixed star"). The fundamental laws of physics do not depend on relative motion nor relative acceleration; they are valid for both inertial frames and noninertial frames of reference. A precursor to this is the recognition by Galileo and Newton that gravitational and inertial mass seem to be the same for all substances.

Weak Field General Relativity: Fairly simple arguments show that we can derive some first order general relativity results just using some of the arguments discussed in all of the preceding text above. We don't need the full power of Einstein Field Equations. Picture in your mind the surface of the Earth with some objects above it which we will allow to fall freely under gravity (and no atmosphere, just ideal vacuum). Consider a clock 'A' placed h meters above clock 'B' in a local gravitational field, g, with another reference comparison clock 'C' lying high but nearby at a fixed altitude [9]. The GR principle says that the physics of this system is equivalent to that where clocks A and B accelerate

upwards with acceleration $a = |g|$. Then, by conservation of kinetic plus potential energy, the speeds of the clocks when they pass altitude C must obey $v_B^2 = v_A^2 + 2ah$. By SR, the clock periods dilate by $T = \gamma\tau \simeq \tau(1 + v^2/2c^2)$. Then period:

$$T_B \simeq T_A[1 + (v_B^2 - v_A^2)/2c^2] \simeq T_A(1 + gh/c^2) \simeq T_A\left[1 + \frac{GM}{c^2 r_B} - \frac{GM}{c^2 r_A}\right]. \tag{19}$$

We have used the approximation $1/r_B - 1/r_A = (r_A - r_B)/r_A r_B = h/\bar{r}^2$ and $GM/r^2 = g$. But we really can't go beyond little h distances to long radial r distances like we do in general relativity.

This period elongation, $T_B > T_A$, is called 'Red Shift.' That is, since the lower time period is longer, the perceived frequency is lower (and we say shifted toward the red). This concept has been proven to apply to both light and to 'matter waves' as well [16]. Phase difference measurements in an atom or neutron interferometer are the same as those accumulated using conventional clocks following the same paths (test of the principle of equivalence, the famous COW experiment [22]). The first accurate test of gravity on photons was the 1960 Pound-Rebka experiment for gamma-rays from Fe^{57} at an elevation of 22.5 meters above a detector (verified with accuracy $\pm 1\%$ [20]). Atomic clocks are now so accurate that a change in time flow should be seen over an elevation change of only 2 cm (Jun Ye, JILA/Nist test in Boulder, [21]). The degree of gravitational redshifting $\Delta\lambda/\lambda \sim \Delta\Phi/c^2$ is just -2 parts-per million for photons leaving the Sun but a powerful $-10\,\%$ for neutron stars [25] where the needed escape speed is 30% of light speed.

A similar comparison exists for measuring rods in the radial direction where now $L = L_o/\gamma \simeq L_o(1 - v^2/2c^2)$. Then,

$$L_B \simeq L_A[1 - (v_B^2 - v_A^2)/2c^2] \simeq L_A(1 - gh/c^2) \simeq L_A[1 - GM/c^2 r_B + GM/c^2 r_A] \tag{20}$$

If A is far away (e.g., the earth observing the sun), then

$$\frac{dt'}{d\tau} \simeq \frac{T_B}{T_A} \simeq [1 + GM/c^2 r] \quad and \quad \frac{dr'}{dr} \simeq \frac{L_B}{L_A} \simeq [1 - GM/c^2 r]. \tag{21}$$

The term GM/c^2 is often shortened to just m. This is especially true when using modern units where basic constants are set equal to unity, $c = \hbar = \ G = 1$. The equations above can be assembled by components into a metric:

$$\mathbf{d\tau^2 \simeq dt'^2(1 - 2m/r) - dr'^2(1 + 2m/r) - dr'_\perp{}^2} \tag{22}$$

which resembles the linearized Schwarzschild metric. But this was only constructed using the principle of equivalence and special relativity for weak fields.

The arguments leading to equation (22) can be reinforced by other physical considerations. Simply by conservation of energy and basic quantum laws, a photon of energy $E = h\nu$ rising against a gravitational potential must have its frequency lowered by $\Delta\nu/\nu = gh/c^2$.

The red-shifting due to the field of our sun is a tiny contribution (e.g., parts per million). Since $\nu\lambda = c$, $d\nu/\nu = -d\lambda/\lambda$. If T is period, and ϕ is gravitational potential $-MG/r$, then [10]:

$$\frac{\nu_A - \nu_B}{\nu} = \frac{T_B - T_A}{T} = \frac{\lambda_B - \lambda_A}{\lambda} = \frac{\phi_B - \phi_A}{c^2} \tag{23}$$

Duration is a number of periods and length is a number of wavelengths. So this result is consistent with the first order length transformation (20). Massive particles also obey $E = h\nu = mc^2$ and will also suffer frequency change from change in gravitational potential.

Also notice that the radial component of the speed of light is no longer seen as constant everywhere,

$$\frac{dr'}{dt'} \simeq \frac{dr}{d\tau}\left(\frac{1 - m/r}{1 + m/r}\right) \quad \Rightarrow c' \simeq c(1 - 2m/r) \tag{24}$$

it slows down in near field. Light speed c is a local constant, but at distance separation it is non-constant and non-isotropic. This slowing down of the apparent speed of light is similar to having the gravitational field act as a refracting medium. Then light rays passing through this medium will get bent. This can be used to derive the "bending of starlight" and the "time delay of radar." So, we can get three of the simpler consequences of testing GRT. With care, the first approximation of the perihelion shift of the orbit of the planet Mercury can also be attained. This means that full testing of the Einstein Field Equations requires strong fields (such as the famous binary pulsar first seen by the giant Aricebo radio telescope [Puerto Rico, 1974]).

Geodesics are world lines of extremal proper time [8]: The solutions for trajectories in general relativity are curved pathways called geodesics. These are like straight lines for light rays in Euclidean space or great circles on the surface of the Earth. We wish to talk about particle paths in the gravitational field just above the surface of the Earth, and we just showed that clock frequency speeds up with height. And in special relativity, we showed that clock frequency slows down with speed. Recall that a little trick here is that frequency is 1/ clock period; they are inversely related. Now we wish to combine these. There is a time flow tradeoff between speed and elevation called "most hang-time and least speed" [Feynman] over desired trajectories. Or, in the Feynman Lectures, it is said, "An object always moves from one place to another so that a clock carried on it gives a longer time than it would on any other possible trajectory – with of course the same starting and finishing conditions" [15]. We can combine the previous math for time dilation (12) and gravity time (19) to get:

$$\left(\frac{d\tau}{dt}\right)^2 = g_{oo} - \left(\frac{dx}{cdt}\right)^2 = g_{oo} - \frac{v^2}{c^2}, \quad or \quad \frac{d\tau}{dt} = \sqrt{1 - \frac{2MG}{c^2 r} - \frac{v^2}{c^2}} = \Gamma^{-1} \tag{25}$$

where Γ could be called a "gravitational Lorentz factor" [23] and u is a velocity accompanying each potential. To low order, one could approximate this as:

$$\frac{d\tau}{dt} = \frac{1}{\Gamma} \simeq (1 - \frac{MG}{c^2 r} - \frac{v^2}{2c^2}), \quad or \quad \Delta\tau \simeq \Delta t \left[1 + \frac{g\Delta h}{c^2} - \Delta\left(\frac{v^2}{2c^2}\right)\right] \tag{26}$$

Since $\omega_t/\omega_\tau = d\tau/dt$, we do see that clock frequency (in perceiver frame) is slowed down by motion and speeded up by height. The fraction $c^2/g = R$ is the relativistic radius of curvature of the Earth's gravitational field (which works out to be about one light year)[23]. Recall in equation (16) under rest mass that frequency $\omega = \gamma E_o/\hbar = \gamma m_o c^2/\hbar$. This now becomes $\omega = \Gamma m_o c^2/\hbar = \omega_o + \Delta\omega$. This suggests multiplying the last equation (26) by $m_o c^2/\hbar$. Then, $\Delta\phi = (-L)\Delta t/\hbar$ where $L = T - V$ is just the simple Lagrangian as in the older equation (11).

Finally, returning to the principle of extremal proper time, the whole proper time accumulated along a trajectory is given by

$$\tau = \int_{t_1}^{t_2} d\tau = \int_{t_1}^{t_2} g_{\mu\mu} dx^\mu dx^\mu dt \simeq \int_{t_1}^{t_2} (1 - \frac{L}{m_o c^2})dt \tag{27}$$

In finding extremum's, added constants and proportional factors don't matter, so we are left with just the usual least action variation of the Lagrangian: $\delta\tau = 0 \Longrightarrow \delta \int L dt = 0$. [24]

Proper time (time carried by a moving frame) τ is a maximum, and $A = \int L dt$ is a minimum. In a way, this principle of maximum proper time is another way to derive the simplest Lagrangian, $L = T - V$ (this time for gravitational potential energy).

Cosmology: Finally, a fair understanding of cosmology can be attained simply by using Newtonian calculations for the case of a homogeneous and isotropic universe with zero curvature ($k = 0$), no cosmological constant ($\Lambda = 0$), no pressure, and only matter (like dust). From simple conservation of energy, equations can be derived resembling Einstein's general relativity field equations. This is essentially the Einstein de Sitter (EdS) model of 1932 for a "just right" universe that barely expands forever. This is discussed in many older references and was a dominant model in cosmology for nearly 50 years [24] [25]. Of course, we are missing the early radiation era of the expansion of the universe which was dominant until 47,000 years after the big bang. And we are also missing the accelerated expansion era which may have begun 7 billion years after the big bang.

This special EdS case is contained in Friedmann equations begun in 1922 which can also be approached using Newtonian conservation of energy [26]. An easy outline of essential

[23]The trajectory of a ball tossed into the air is a parabola. Change the time axis to ct, and this parabola is the approximation to the top of a really great circle of radius R (found simply by calculating radius of curvature from standard calculus formula). So, $g\Delta h/c^2 = \Delta h/R$ is a really tiny number.

[24]Feynman's derivation of this is somewhat easier than mine, see Vol II pg 42−13.

[25]Despite its historical importance, EdS is not now dominant in current books on cosmology or general relativity. Its primary utility was easy integrations of its equations for applications.

equations for Newtonian cosmology is contained in [27]. Of course, ultimately, one would wish to know the proper study of general relativity using the concept of curvature. And GRT says that the previous understanding of Newtonian gravitation is conceptually wrong and should be stated as a low order curvature of time, $d\tau/dt = g_{oo}(r)$. Beyond that, bending of light (traveling at the speed of light) sees an additional but equal contribution due to the curvature of space. This doubles the bending that Newton might have predicted.

A Newtonian approach can also be used to easily understand cosmic inflation too. Imagine an ideal case of a ball freely falling through cylindrical hole drilled all the way through the center of the Earth. Remember that the acceleration of gravity, g, only depends on the mass contained within a spherical shell at radius r. Without any air resistance, the motion is approximately that of simple harmonic oscillation with a period of 1.4 hours. Now switch from gravity to anti-gravity from the cosmological constant, Λ to give $F = +k\,r$ with $\Lambda \sim -8\pi G\rho/c^2$ in its behavior. Its solution now changes from sine-wave motion to rapid exponential expansion like $r = r_0 e^{+\sqrt{\Lambda/3}\,t} = r_o\, e^{Ht}$ [24].

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6. Appendix

More Traditional Derivation of Relativistic Length Contraction:

The usual textbook calculation of length contraction is a little longer than the simple argument shown previously above [10]. It depends on a light flash mirror bounce from the end of a rigid moving bar. So imagine a longitudinal bar (say in a moving or primed S' system) moving to the right along an x-axis. A light flash is sent from the left of the bar to a mirror, M, on the right side which reflects the light back to the left end for a round-trip journey. The initial light flash event is at initial moving time $t' = t = 0$ for moving clocks versus clocks at rest in an un-primed system S. We compare length and time for the events that (0): initial flash, (1): light bounces from M, and (2) light received at the left end of the bar at final times t and t'. The length of the bar is $L' = ct'/2$; and the times recorded in the rest system are t_1 and t_2.

Since S' is moving, the time at which light hits mirror M is $t_1 = (L + vt_1)/c$. Or $t_1 = L/(c - v)$. And then the time back to O' is short because the left end of the bar has moved a total distance $x_2 = vt_2$ during round-trip transit.
Consider the last time increment $(t_2 - t_1) = (x_1 - x_2)/c$ where final $x_2 = x_1 + v(t_2 - t_1) - L$ or $(t_2 - t_1)(c + v) = L$. Then,

$$t_2 = (t_2 - t_1) + (t_1 - 0) = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2Lc}{c^2 - v^2} = \frac{2L/c}{1 - \frac{v^2}{c^2}} = \frac{2L\gamma^2}{c}. \tag{28}$$

Now remember from equation (12) that the moving clock time is $\tau = t' = t/\gamma$, and $t = t_2$, and $L' = ct'/2$, so,

$$L = \frac{tc}{2}\left(1 - \frac{v^2}{c^2}\right) = \frac{tc}{2\gamma^2} = \frac{t'c}{2} \cdot \frac{1}{\gamma} = \frac{L'}{\gamma} \tag{29}$$

So, the length of a moving rod is seen as contracted by the system at rest, $L = L'/\gamma$. Notice that we have had to use the same speed of light factor c in both the S system at rest and in the moving system S'. So Einstein's second postulate of the constancy of the speed of light is still required.

Free Fall: Technically, it is not quite true that free fall in a gravitational field is the same as the effects of an observer's acceleration [12]. Real gravitational fields have tidal forces so that the Riemann tensor is non-zero. In Newtonian gravitation, tidal accelerations mean that objects at different altitudes experience different relative accelerations, $\Delta a \simeq 2MG\Delta h/R^3$. Tidal accelerations cause divergence of initially parallel geodesics in the curved space-time of GR. The equivalence principle was a guiding concept towards GR but acted as a midwife rather than actually constituting an explicit portion of GR. Nevertheless, it could be argued that PE combined with SR should produce space contraction along with red-shifting time effects and that the Schwarzschild form of a metric tensor is more physically valid than an isotropic form. The principle of general covariance would argue otherwise; but it really doesn't have a legitimate power to be convincing. For an external observer 'relatively' lacking in velocity with respect to a central mass, the radial coordinate about the central mass is 'really' different from the angular coordinates because of radial spatial contraction. And radial space contraction and time dilation only need to be approximated to first order in gravitational potential to yield the correct perihelion shift [11].

The First Test of Gravitational Red Shift: Notice that the term $-GM/r$ is just Newtonian gravitational potential, φ. In weak fields and negligible speeds, $dt/d\tau \sim 1/\sqrt{1+2\varphi/c^2} \sim 1-\varphi/c^2 = \nu_o/\nu$. If ν is light frequency (the inverse of light period), $\nu(r) \sim \nu(r_o)(1+\Delta\varphi/c^2)$. On the surface of the earth, $\nu(h) = \nu(h_o)(1 - g(h - h_o)/c^2$). This important 'red shift' of light at different potentials has been verified experimentally even over short altitude changes on Earth [e.g., within $\pm 1\%$ for the 'Pound-Rebka' experiment over $\Delta h = 22.5\, m$ back in 1959 [8]] [26].

Some comments on going Beyond Non-Relativistic QM to Quantum Field Theory (QFT)

A main difference between QM and QFT is that at higher energies, the number of particles present is not conserved. Matter and radiation are easily inter-converted (as long as appropriate quantum numbers are preserved). Key new operators are then introduced beyond those of relativistic QM: creation operators and annihilation operators (called $\hat{A}^\dagger$ *and* $\hat{A}$) and are related to the raising and lowering operators for the energy levels of the Linear Harmonic Oscillator (LHO). This is very different from ordinary (non-relativistic) quantum mechanics where we discuss the evolution of a "particle" already in existence with particle number being held constant. Psi is not a probability amplitude but

[26] Actually, weak field red-shift can be derived without General Relativity by simply using the principle of equivalence and special relativity (see Schiff [9]).

operators which create and destroy particles in various normal modes.

Fundamental reality is composed of fields. And there are two basic types called "matter" fields and "interaction" (or "gauge") fields, and they have quanta for fermions and bosons (half-integral spins and integral spins). Fundamental interactions occur only between matter and interactions fields [7]. "So, is QFT really based more on particles or on fields? Although there is still a little disagreement, a strong majority of theoreticians favor fields as fundamental objects. Nature is made of fields. Quantum fields permeate space-time, are relatively eternal and omnipresent, and have excited state quanta that we have traditionally called 'particles.' There is a special quantum field for each type of elementary particle.

Matter in general is an excitation or wave in one or more of the fermionic matter fields. For an electron two-slit diffraction for example, the extended singly-excited electron field goes through both slits. The interaction with a detector screen is deduced to have been from a 'particle.' "Although excitations belong to the entire field, they must interact locally." Of course, there is a problem with the word "field" in QFT (or any other classical word used to describe quantum mechanics). It is usually defined as having a value (e.g., scalar or vector, etc.) assigned to every point of space-time. We picture that simplistically as an amplitude disturbance in a mattress of springs. But the field in QFT is much more "magical" than that. Many different types of disturbances can occur at the same time in a given place and be holistically coordinated with all other locations.

The central problem with a particle interpretation is that the primary attribute of a particle should be its localization in space, and particles should be countable. But there is no such thing as an observable for position in QFT, and Wigner said in 1973 that every attempt to provide a precise definition of a position coordinate stands in direct contradiction to relativity. A 'photon' is not localizable at all, not even approximately, and there is no consistent space-time wave-function for a photon as a "particle." For single photons, one can think of an electromagnetic wave packet as a function of space-time. In general, there is no accepted viewpoint on the subject of localization in QFT that is either simple or clear even for the case of free fields. Peierls said (1973) that "at relativistic energies, the electron shows the same disease. So in this region, the electron is as bad a particle as the photon." Quantum fields are intrinsically delocalized and unbounded,?