Notes from An Introduction to Modern Cosmology, by Andrew Liddle

The cosmological principle, p. 1-2.

The cosmological principle is an extension of the evolving scientific view that we and our surrounding area are not special. The Mediterranean Sea, Earth, the solar system, and the Milky Way galaxy are not the center of the Universe. If you ignore "small" irregularities, (out to about the scale of galaxy clusters) the Universe is about the same everywhere (homogeneous), and in every direction (isotropic). This large-scale approximation holds quite well, as far as can be observed. On a scale smaller than galaxy clusters, it breaks down because matter is not uniformly distributed. (Note that being isotropic everywhere implies homogeneity, but not vice versa. Consider an infinite stack of paper: It is the same everywhere, but directions in the plane of the sheets are different than directions that cross multiple sheets.) The assumption that all places and directions are the same is fundamental to modern cosmology.

Expansion of Universe, recession velocity, Hubble "constant", p. 9, 21, 33, 45, 57.

The Universe is expanding, apparently even accelerating. Any two points in the Universe that are far apart (not bound together by gravity or some other force, i.e. beyond the scale of galaxy clusters) are receding from each other at a velocity proportional to the distance between them. The Hubble "constant" is the proportionality constant. It is believed to be constant across all of space, but it varies with time as the expansion rate of the Universe changes. It is defined as $H(t) = \frac{v}{r}$, where v is the velocity and r is the distance. Its value at the present time $t_0 = 13.7$ billion years, the age of the universe, is $H_0 = H(t_0) = 72 \pm 8$ Km/sec/Megaparsec (2007). This is often represented as $100h$, where $h = .72 \pm .08$, to accommodate uncertainty in its value. After correction for dissimilar distance units, its units are time^{-1}, or fractional change per unit time. Numerically, this says space is presently expanding at approximately 73 parts per trillion per year, or 3/4 of an atomic diameter per meter per year. It also says that if the Universe had been expanding at the same rate throughout its life, it would reach its present size at about 13.6 billion years of age. It hasn't been, and the closeness of this Hubble time to the age of the Universe is somewhat coincidental. Note also that ordinary matter on Earth is not expanding at this rate because it is held together by electrical and gravitational forces. This is the rate of expansion of space itself, not galaxy clusters or other, smaller concentrations of matter (p. 21). Since velocity is proportional to distance, two distant points can recede from each other at greater than the speed of light. This is allowed because neither of them is moving through space faster than light, they do not encounter each other with excess relative velocity, and no signal can pass between them. Only space itself is moving faster than light (p.21).

Scale factor of the Universe, comoving coordinates, p. 19.

In dealing with the expansion of the Universe, distances may be specified as physical distances, or as differences of "comoving coordinates", which expand with space. The distance between two points in comoving coordinates does not change as space expands, while the physical distance does. The scale factor of the Universe, a(t), is defined as $a(t) = \frac{|\vec{r}|}{|\vec{x}|}$, where $|\vec{r}|$ is physical distance and $|\vec{x}|$ is comoving distance. Since the Hubble parameter is the ratio of physical velocity to distance for any two points, it is often expressed in terms of a, $H(t) = \frac{v}{r} = \frac{\dot{a}(t)}{a(t)}$. ($\dot{a}$ means $\frac{da}{dt}$.) In a homogeneous Universe, a depends only on time. It is believed that the Universe has been expanding continuously throughout its life, and may continue forever. Therefore, $\dot{a}$ is always positive and $a(t)$ increases monotonically.

Redshift, p.9, 34, 125.

Although the expansion of space does not stretch matter or clusters of matter held together by gravity, electrical or nuclear forces, it does stretch photons. Redshift z is the fractional increase in wavelength λ due to the expansion of space between emission and observation.

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{\lambda_{obs}}{\lambda_{em}} - 1.$$

The ratio of wavelengths before and after stretching, $z + 1 = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})}$, is exactly the ratio of scale factors of the Universe at the start and end times of the path. In effect, the expansion of space stretches the wavelength of photons along with it. Due to the uniqueness of the speed of light, and the assumed monotonicity of a(t), specifying the redshift of presently observed light also uniquely specifies the time and distance of its emission.

Cosmic microwave background, black body radiation, p. 12-15, 75-6.

About 350,000 years after the big bang, the Universe was about .001 of its present size, and the temperature had dropped to about 3000K. At that temperature, matter glows brightly with a yellowish white color. Above that temperature, hydrogen, which made up most of the Universe, is ionized. The thermal radiation photons are interacting rapidly with the free electrons in the plasma, and are not able to travel large distances. As the temperature drops, the hydrogen ions capture the electrons, forming neutral hydrogen atoms. The thermal photons are suddenly able to move freely, since they do not interact with the atoms. Because the density of the Universe is very low, they can travel very long distances, and are in fact still detectable, arriving at Earth from all directions. This is the cosmic microwave background.

This radiation, which started out about the color of an ordinary incandescent light bulb, has been redshifted by the expansion of space, from yellowish to reddish to infrared, and all the way to microwave. Its wavelength is about 1000 times as long as it was, but the original and final spectra both have the form of black body radiation, initially around 3000K, and now at 2.725K.

The energy density spectrum of this radiation is the expression usually used to describe it. This spectrum has a broad peak at an energy of about 3kT, and a similar average photon energy. Integrating the density spectrum over energy to get total energy density at the present time shows that the total density is $\varepsilon_{CMB} = \alpha T^4$, where $\alpha = 7.565 * 10^{-16} Jm^{-3} K^{-4}$. (Numerically, this is about 42 Joules, a small firecracker, in a 100 Km cube. Or enough energy in each cubic kilometer to lift a small drop of water a millimeter.)

Once the thermal radiation has "decoupled" from matter (because the temperature has dropped too low for them to interact), it travels freely through space, affected only by being stretched along with the scale factor. Since the radiation energy density decreases inversely as the fourth power of scale factor (3 for volume, 1 more for stretching of the wavelength of each photon), we must have $T \propto \frac{1}{a(t)}$, the radiation temperature inversely proportional to scale. This preserves the characteristic shape of the black body radiation spectrum while shifting it to lower temperature. It also explains how the temperature falls from about 3000K to 2.725K as the size of the Universe increases about 1000 times since decoupling.

Expansion of the Universe - Mathematical description

The expansion of the Universe is described by three equations, the Friedmann equation, the fluid equation, and the acceleration equation (sometimes called one of the Friedmann equations). These are not independent, since the acceleration equation can be derived from the other two. Together, they form a set of differential equations that describe the evolution in time of the scale factor of the Universe, based on what is in it. The cosmological principle is extensively used in the assumption that no point and no direction is unique. This allows the convenience of arbitrary choice of coordinate origin and use of spherical coordinates with dependence only on r.

The principal variables are:

t, time since the Big Bang,
H(t), the Hubble parameter,
a(t), the scale factor of the Universe,
$p_{\_\_\_}(t)$, pressure of various components of the Universe, and
$\rho_{\_\_\_}(t), \varepsilon_{\_\_\_}(t)$, mass or energy density of various components of the Universe.
These are all functions of time and (presumably) not of position. The subscript zero refers to the present value.

k, the curvature parameter, is selectable for positive, negative, or zero curvature, but doesn't change with time or position (p. 20).
Λ, the cosmological constant, is given by nature and is presumed not to change with position. It may change with time in an unknown manner.

The Friedmann Equation, p. 18-21, 24, 30, 34, 51-53.

The standard form of the Friedmann equation, scaled so that c=1 and a_0 =1, is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}.$$

It can also be scaled so that k=0 or ±1, instead of a_0 =1, but Liddle does not do this. This equation describes how the scale factor changes in time, depending on how the density in the Universe changes, as well as its curvature.

The fluid equation, p. 22.

To use the Friedmann equation, the behavior of density $\rho(t)$ over time must be specified. The fluid equation relates the time behavior of density, scale, and pressure:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0.$$

Pressure can be eliminated from this equation with the assumption that it is a unique function of density (the equation of state), determined by the type of matter or energy in the Universe.

The acceleration equation, p. 23.

The acceleration equation can be derived from the other two, and describes the acceleration of the scale factor:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}.$$

This equation also needs the equation of state to eliminate p. Note that p and ρ have the same sign. Density and pressure both have the same sign, so they both decelerate expansion. Counterintuitively, this pressure is not a force pushing the Universe to expand. It is a form of stored energy, and hence an attractive influence, just as mass is.

Solutions

These equations can be simply solved for various special cases in which one or another component of the Universe is dominant. These solutions are useful because the various densities of components of the Universe change monotonically as it expands. This means there are long periods when one is dominant, then a transition to dominance of another.

Radiation dominated, $k \approx 0$, $\Lambda \approx 0$.

In the early Universe, before about 40,000 years, the density was dominated by radiation. (The exact time is determined by measuring the relative abundance of matter and radiation, and extrapolating densities backwards, with photon energy increasing as wavelength shrinks with a.)

The equation of state for radiation gives a pressure $p = \frac{\rho c^2}{3}$ Solving the fluid equation shows that $\rho_{rad} \propto a^{-4}$. The Friedmann equation then gives the following functions of time (remembering that subscript zero refers to the present time and that $a_0 = 1$):

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}},$$

$$\rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0{}^2}{t^2}, \text{ and}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{2t}.$$

Matter dominated, $k \approx 0$, $\Lambda \approx 0$.

Non-relativistic matter has an equation of state $p = 0$, no pressure. The fluid equation then gives $\rho_{mat} \propto a^{-3}$. In a Universe containing both matter and radiation, eventually $\rho_{rad} < \rho_{mat}$, and the expansion will be dominated by matter. The solutions then become:

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}},$$

$$\rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0{}^2}{t^2}, \text{ and}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3t}.$$

In a Universe dominated by matter or radiation, but with no curvature or cosmological constant, the expansion continues forever at ever slower rates.

Negative curvature dominated, k<0, $\Lambda \approx 0$.

The curvature of the Universe is small, but possibly non-zero. Therefore, it does not become dominant until long after matter has taken over from radiation. However, since the ρ terms of the Friedmann equation decrease at least as fast as a^{-3}, while the curvature term decreases as a^{-2}, any curvature eventually dominates. For negative curvature, the Friedmann equation shows that the expansion velocity is constant, $a(t) \propto t$, and expansion continues forever without slowing down ("free expansion").

Positive curvature dominated, k>0, $\Lambda \approx 0$.

If curvature is positive, the right-hand side of the Friedmann equation eventually reaches zero and expansion stops. Assuming that the Universe still contains matter or radiation, its attractive force causes contraction. As the equations are time-reversible, the contraction reverses the expansion function in time.

Cosmological constant dominated, Λ>0.

Although long thought to be zero, the recently discovered acceleration of expansion of the Universe strongly suggests a positive cosmological constant. The cosmological constant can be thought of as the energy density of empty space. If it is positive, it functions like mass and energy density, but with negative pressure. Since its density and negative pressure do not decrease with expansion, the Friedmann and acceleration equations show that eventually this leads to accelerating expansion, as other terms decrease with increasing a(t). The extra term greatly increases the possibilities for behavior of the Universe. In addition to accelerating expansion in a large, low-density Universe, a brief period of inflation in the very early Universe may have been caused by a transient spike in the cosmological constant.

The density parameter, p.47, 52, 100

Looking at the Friedmann equation, one can see that, with k=0, certain combinations of ρ and Λ can balance the equation for any value of H (all given at the same time, usually the present). Or, momentarily considering $\Lambda = 0$, some value of ρ works. This is called the critical density, $\rho_c(t)$. It is a special value because k does not change with time, so a Universe with that density is forever flat. Since ρ is the sum of matter and radiation densities, and Λ can also be expressed as a density, the sum of these densities equaling the critical value yields a Universe where k is always zero. This is useful because observationally, $k \approx 0$, so we know that $\rho_{total} \approx \rho_c$. In addition, with appropriate scaling each of the three quantities ρ_{mat}, ρ_{rad}, or ρ_Λ alone can force k=0 for all time. (With no subscript, the sum of matter and radiation is usually intended.) Thus, for each type of density, there is a ratio of actual to critical value. This is called the density parameter, and is defined for the various densities as $\Omega_{\_\_\_} = \frac{\rho_{\_\_\_}}{\rho_c}$ (all are functions of time). A value of one for the total density parameter gives a solution of the Friedmann equation (with k=0) for the value of H and the densities given. This solution preserves k=0 and $\Omega_{tot} = 1$ for all time. If $\Omega_{tot} \neq 1$, the error increases monotonically, i.e. a flat Universe is unstable; any curvature increases. Therefore, the Universe must have been very nearly flat early on to be anywhere near flat now.

Correction of observational parameters for redshift, distance, curvature, etc.

Luminosity distance, p. 128.

Luminosity distance is the apparent distance to an object, based on radiant energy received from it, with the assumption of $1/r^2$ dependence. In fact, this dependence only holds for small distances. The received energy decreases faster than that for two reasons: The energy of individual photons is reduced as the expansion during travel stretches the wavelength, and the time (distance) between photons is increased for the same reason. Each of these effects reduces the energy rate as a^{-1}, so together they reduce it by a^{-2}. This reduction of observed luminosity increases apparent distance in proportion to the scale factor. Thus $d_{lum} = (1+z)d_{phys}$.

Angular diameter distance, p. 132.

Similarly, angular diameter distance is the apparent distance to an object of "known, fixed" size, based on its subtended angle (the angular extent between two points on the object, as viewed), with the assumption of 1/r dependence and Euclidian geometry. Light from the angular extremes of the distant object propagate toward the observer, preserving the angle between them. Thus, the apparent physical size increases in proportion to the physical distance. This relative increase in subtended angle is equivalent to a reduction in apparent distance, so $d_{diam} = \frac{d_{phys}}{1+z}$. This means that distant objects appear closer than they really are. In fact, their subtended angle only decreases out to about z=1, beyond which objects of fixed size subtend ever larger angles. Thus, the apparent distance not only decreases, but the object eventually fills a larger and larger part of the sky, as its physical size occupies a larger portion of the Universe at the time of emission. The combination of increased angular extent with decreased luminosity at great distances makes observation still more difficult. Not only are the photons lower energy and spread out more in time, but they are spread out over a larger area of the sky as well.

Observationally derived data

Red light, 1.8 eV, 700 nm.
Violet light, 3.1 eV, 400 nm.

Age of Universe ≈ 13.7 * 10^9 years ≈ 4 * 10^{17} seconds.
1 year = 31.5 * 10^6 seconds.

Density of the Universe ≈ 1 galaxy per cubic megaparsec (Mpc), p. 47,63
≈ 10^{11} solar masses/Mpc^3.

Mass of the visible Universe ≈ 10^{11} to 10^{12} galaxies, weighing ~10^{11} solar masses each, p. 47,63
≈ 10^{79} nucleons
≈ 6 nucleons/cubic meter, close to the critical density (see below).

Expansion rate of the Universe
H_0 ≈ 72±8 km/sec/mpc (2007, units are (fraction of length)/time), p. 46
(=100h where h≈.72±.08, to accommodate uncertainty in H_0)
≈ 3/4 atomic diameter/meter/yr
≈ 73 parts per trillion per year.

Constituents of the Universe at t_0, p.63-69, 116:
Critical density ≈ 10^{-29} g/cm^3 = 10^{-29} ρ_{water}
≈ 6 nucleons/m^3 (of which <6% is actual baryonic matter).

$\Omega_{Luminous baryonic matter} \leq .01$ (stars)

$\Omega_{Dark baryonic matter} \approx .03 - .05$ (dust, gas)

Total baryonic matter ≈ 1 proton in five cubic meters.

$\Omega_{Dark matter - non-baryonic} \approx .25 - .30$ (?)

$\Omega_{Cosmological constant} \approx .70$

$\Omega_{Radiation} \approx .00005$ Much less energy density than baryonic matter, but about 10^9 as ,many particles, i.e. 1 photon in five cubic millimeters.

$\Omega_{Relativistic matter} \approx .00003$

$\Omega_{Total} \approx 1.00 - 1.04$, close to critical density, close to flat.