Notes on *An Introduction to Modern Cosmology*

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(with additional material from other sources)

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1 Preliminary Concepts

1.1 Large-Scale Characteristics of the Universe

The ***cosmological principle*** is the unproven but well-supported assumption that, ignoring local variations (which can be huge), any place in the universe is no more special than any other place. This is equivalent to saying that the universe is spatially both ***homogeneous*** (looks the same at each point) and ***isotropic*** (looks the same in every direction). Homogeneity does not imply isotropy, but *universal* isotropy does imply homogeneity. The cosmological principle is obviously not true at the scale of the solar system, the galaxy, or even galaxy clusters, but, making this assumption for the universe as a whole has profound and powerful cosmological consequences.

It would seem that the expansion of the universe violates the cosmological principle, since we appear to be in a privileged location at the center of the expansion, with expansion rates proportional to distance away from us. This does not violate the cosmological principle, though, since an observer at any point in the universe would see the expansion as centered on them with the same proportionality between recession rate and distance that we see.

The ***perfect cosmological principle***, the idea that the universe is both spatially homogeneous and isotropic as well as homogeneous in time, was the basis of the ***steady-state theory*** and appears to be almost certainly wrong.

1.2 Redshift, *z*, and Hubble's Law

The ***redshift*** of spectral lines is defined observationally as $z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}$. Redshift is related to the magnitude of the recessionary (radial) velocity of a galaxy, u, as $u = z\,c$. A galaxy's recessionary velocity is composed of two parts; a radial velocity due to expansion of the universe and a randomly oriented ***peculiar velocity*** due to gravitational perturbations induced by other nearby matter. The peculiar velocity makes Hubble's Law, $\mathbf{v} = H_0\,\mathbf{r}$, approximate, with the approximation being worse for nearby galaxies.

The proportionality constant, H_0, is called the ***Hubble parameter*** (or ***Hubble constant***, though it's not actually constant ($H = H(t)$) over cosmological time). The zero subscript in this symbol, and other symbols in his document, refers to the value of that quantity at the present time. Note that the assumption of isotropy requires that $H \neq H(\theta,\ \phi)$, and so expansion is purely radial. Measurements of H_0 are notoriously difficult and their uncertainty is parameterized by h (not to be confused with Plank's constant), as:

$$H_0 = 100\,h\,\text{kg}\cdot\text{sec}^{-1}\cdot\text{Mpc}^{-1} = \frac{h}{9.8\times10^{9}\ \text{yr}} = 2.13\,h \times 10^{-33}\ \text{eV}\cdot\,h^{-1} \text{ with } h = 0.72\ \pm 0.02.$$

The above expression for H_0, while useful when thinking of it as a measure of the recessionary velocity of the expanding universe, obscures another, equally important way of thinking about the Hubble parameter. Changing the units gives, $H_0^{-1} = 9.77\, h^{-1} \times 10^9$yrs, called the ***Hubble time***, a crude estimate of the age of the universe - crude because it ignores changes in the expansionary velocity over the history of the universe.

■ 1.3 The Scale Factor, *a(t)*

The time-dependent Hubble "constant" is defined in terms of the ***scale factor***, $a(t)$, as $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$. The scale factor measures how physical dimensions change with time. The redshift is a natural consequence of the Doppler effect which stretches wavelengths as, $\lambda_{\text{obs}} = \lambda_{\text{em}}\, a(t_{\text{obs}})\ / a(t_{\text{em}})$. Combining this with the above and relabeling $a(t_{\text{obs}}) = a_0 = 1$ gives,

$$z = \frac{1-a(t_{\text{em}})}{a(t_{\text{em}})} \text{ or } a(z) = \frac{1}{1+z}.$$

■ 1.4 Comoving Coordinates

Because general relativity (GR) requires that the laws of physics are the same in any coordinate system, including non-inertial ones, we can choose to work in any convenient coordinate system. We will often choose to use ***comoving coordinates***, $\mathbf{x}$, that are carried along with the universal expansion, rather than ***physical coordinates***, $\mathbf{r}$, which define a fixed coordinate system in which the expansion takes place. These physical coordinates are those we live in on earth (and anywhere within the galaxy) and those of any bound system, that are not affected by the expansion of the universe.

The transformation between these coordinate systems is given by, $\mathbf{r} = a(t)\,\mathbf{x}$. Note that, for objects that are stationary relative to the observer, $\dot{\mathbf{x}} = 0$, but $\dot{\mathbf{r}} = \dot{a}(t)\,\mathbf{x}$. Note also that our assumption of isotropy is only true for comoving coordinate systems.

The ***comoving horizon*** is defined as the surface of a sphere centered on the observer whose radius is the distance light could have traveled in the absence of interactions since the origin of the universe. It defines the observable universe. Due to the expansion of the universe, its radius is not simply the age of the universe times the speed of light.

To locate the comoving horizon, we need another concept called the ***conformal time***, η. In time interval $d\,t$, light in an expanding universe with scale factor $a(t)$ will have traveled a distance $d\,\eta = \frac{d\,x}{a} = \frac{c\,d\,t}{a}$. Setting $c = 1$, we write, $\eta \equiv \int_0^t \frac{d\,t'}{a(t')}$. The conformal time is not any physically meaningful time, but η_0 is the time it would take a photon to travel from our current location to the edge of the observable universe, if the universe were to suddenly stop expanding. The comoving horizon is at a distance equal to c times the conformal time.

■ 1.5 Curvature, *k*

The cosmological principle demands that space must exhibit the same overall curvature at every point. Local variations in curvature, just like local variations in ***mass-energy density***, ρ, are of course, tolerated. There are only three spatial geometries that allow constant curvature: ***flat*** space, uniformly positively curved (***closed***) space, or uniformly negatively curved (***open***) space. We will address curvature in more detail in the section on GR, but for our purposes now, we can capture what we need of curvature in a single number, the ***curvature parameter***, k. Curvature is determined by the mass-energy density of the universe. There is a particular value of mass-energy density, called the ***critical density***, ρ_c, that must exist for the curvature of spacetime to be flat. Universes with $\rho < \rho_c$ are open, while those with $\rho > \rho_c$ are closed.

We can force k to take on one of three discrete values: zero for flat space, +1 for closed space and -1 for open space. Although we can't draw 3-dimensional curved spaces, we can get a feel for their behavior by considering curved 2-dimensional surfaces.

For the "flat" (or Euclidean) space we are used to, geometric figures behave the way we were taught in school. The shortest distance between two points is a straight line. Flat universes are infinite, since, if they had an edge, the cosmological principle would fail there. No points in flat space are any more "special" than any other point.

A non-rotating 2-dimensional spherical surface also has no "special" locations. If its radius is finite, it has no edge, but has a finite area, $4\pi r^2$. Geometric figures on a sphere behave strangely, with the angles of a triangle adding up to more than 180° and the circumference of a circle being less than $2\pi r$. "Straight lines" don't exist on a spherical surface and the shortest distance between points is the arc of a ***great circle*** - a circle that defines a plane that passes through the center of the sphere. Great circles in a spherical geometry are examples of the more general concept of a ***geodesic*** - the curve that defines the "shortest distance" between two points in any space, regardless of its curvature.

Negative curvature is usually modeled in 2-dimensional space by a saddle-shaped surface. This not a perfect analog for the 3-dimensional version, since it's not possible to construct a 2-dimensional surface of uniformly constant negative curvature in 3-dimensional space. The saddle-shaped surface will only have constant curvature at the center point of the saddle. Thus, the center point is unique, meaning that it also violates the cosmological principle. Nevertheless, the approximation shows that the angles of a triangle in negatively curved space add up to less than 180° and the circumference of a circle is more than $2\pi r$.

1.6 The Energy-momentum Relation

The energy of any particle is given by Einstein's famous $E = m c^2$ when we understand E to be the particle's total energy and m to be its ***relativistic mass***, composed of its ***rest mass***, m_0, and the additional mass that results from its velocity. A more useful version of this equation, known as the ***energy-momentum relation***, expresses how these two components combine to form the particle's total energy by expressing E as a function of the rest mass and the magnitude of the particle's momentum, p,

$$E^2 = m_0^2 c^4 + p^2 c^2.$$

Important special cases of this relation are:

1) ***Massless particles*** ($m_0 = 0$) such as ***photons***: These relativistic particles have an energy associated only with their motion, $E = c\,p$. This expression for radiation energy was extended in the quantum theory as (taking $c = 1$), $E = h f = \frac{h}{\lambda} = \hbar\,\omega$. Combining these equations gives the de Broglie relation, $\lambda = \frac{h}{p}$.

2) ***Massive relativistic particles***: These are particles, such as cosmic rays, that are moving relative to us at speeds, u, approaching c. They can never move at exactly the speed of light, since then their momentum, $p = \gamma\, m_0\, u$, and hence their energy, would become infinite ($\gamma = \frac{1}{\sqrt{1-(u/c)^2}}$). Both terms in their energy-momentum relation are significant contributors to their total energy.

3) ***Non-relativistic particles*** ($u \ll c$): The first two terms of the Taylor expansion of the energy-momentum relation for small u gives $E \cong m_0 c^2 + \frac{p^2}{2 m_0}$; simply the sum of the particle's rest mass energy and its classical kinetic energy (recall that classically $p = m_0\, u$, so $K.E. = \frac{p^2}{2 m_0} = \frac{1}{2} m_0 u^2$).

1.7 The Fundamental Constituents of the Universe

There are several broad categories of constituents of our universe.

1.7.1 Baryons

Baryons are usually thought of as particles composed of 3 quarks, but cosmologists also lump mesons (composed of 2 quarks) and electrons (a type of lepton - not quark-based at all) under this heading. The justification for this seems to be that, of these particles, only protons and neutrons (both baryons in the traditional sense) contribute in any significant way to the mass of the universe. Since charge neutrality arguments mean that there must be the same number of electrons as protons, and protons are almost 2000 times more massive, the mass of the electrons in the universe is swamped by that of the protons. All other baryons are unstable and decay into protons and neutrons, so they also can be ignored. Thus, when cosmologists say “baryons,” what is meant is nucleons - protons and neutrons.

In the present universe, baryons are typically moving much less than light speed, so they fall into the non-relativistic category (3) above.

1.7.2 Radiation

By “radiation”cosmologists mean *electromagnetic* radiation - photons - and do not include α, β, or neutron radiation. The ionizing properties of energetic photons are especially important cosmologically, so it may be useful to review the physics of electromagnetic radiation.

In systems that are at ***equilibrium***, some quantity is exchanged between the elements of the system in such a way that all ***microstates*** (configurations of its constituents) available to the system are equally probable. The science of ***thermodynamics*** studies several types of equilibria - each with its own exchanged quantity. These are shown below.

Equilibrium type	***Exchanged quantity***
thermal	energy
mechanical	volume
diffusive	matter

All of these equilibria are important in cosmology, but we will focus here on thermal equilibrium among photons. Because they are quantum mechanical particles, photons obey ***quantum statistics -*** the study of dense systems in which two or more identical particles have a reasonable chance of trying to occupy the same single particle state. Since photons are ***bosons*** (integer spin; spin-1 for photons), an unlimited number of them can occupy the same state.

In most gases with which we are familiar, the inter-particle distance is large compared with the particle’s de Broglie wavelength (so that the wave functions of the particles in the gas do not overlap), but in the very dense conditions of the early universe, this simplifying assumption breaks down. It can be shown that, under these conditions in which quantum effects become important, the ***occupancy number*** of the i^{th} state (average number of bosons in the state) is,

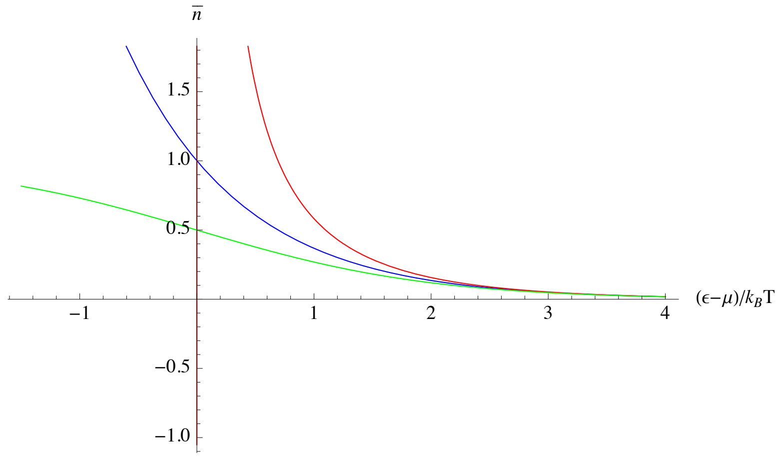
$\overline{n}_i^{\text{BE}} = g_i \Big/ \left(\exp\left(\frac{\epsilon_i - \mu}{k_B T}\right) - 1\right)$, where g_i is the degeneracy of the state, ϵ_i is its energy, μ is the chemical potential, $k_B = 8.6 \times 10^{-5}\ \text{eV} \cdot K^{-1}$ is Boltzmann’s constant and T is the temperature in Kelvins.

Aside on Quantum Statistics

The above expression for $\overline{n}_i^{\text{BE}}$ gives the occupation number for bosons that obey ***Bose-Einstein statistics***. ***Fermions*** obey ***Fermi-Dirac statistics***, with an occupation number, $\overline{n}_i^{\text{FD}}$, that differs only by a *+1* in the denominator rather than the *-1*. There is a third distribution (with no accommodation for degeneracy and no additive constant in the denominator) called the ***Maxwell-Boltzmann distribution*** that represents the non-quantum limit of the other two. There are no physical situations in which particles behave with exactly a Maxwell-Boltzmann distribution, but it’s a useful approximation for diffuse gases. It’s interesting to see how these distributions (assuming a degeneracy of 1 for BE and FD) look as a function of $(\epsilon - \mu)/k_B T$. As the exponent gets large, the distributions merge. The curves are:

Bose-Einstein, $\overline{n}_i^{\text{BE}}$
Maxwell-Boltzmann
Fermi-Dirac, $\overline{n}_i^{\text{FD}}$

```
Plot[{1 / Exp[x], 1 / (Exp[x] - 1), 1 / (Exp[x] + 1)}, {x, -1.5, 4},
 AxesLabel → {"(ϵ-μ)/kBT", n̄}, PlotStyle → {Blue, Red, Green}]
```



The states for photons are their allowed vibratory modes. With $g = 1$, $\epsilon = h\,f$ and $\mu = 0$, the occupation number distribution for photons is

$$\bar{n}_P = 1\big/\left(e^{h\,f/k_B\,T} - 1\right).$$

This is known as the ***Planck distribution***. It was the solution Max Planck offered for the "ultraviolet catastrophe," the classical prediction that, with all frequencies allowed, the energy in the electromagnetic field should be infinite. With the Planck distribution, the energy contribution of frequencies much greater than $k_B\,T/h$ is exponentially suppressed.

To compute the total EM energy emitted by a radiating body, we must integrate this function over all modes. For a radiator of length L, the allowed wavelengths and momenta are,

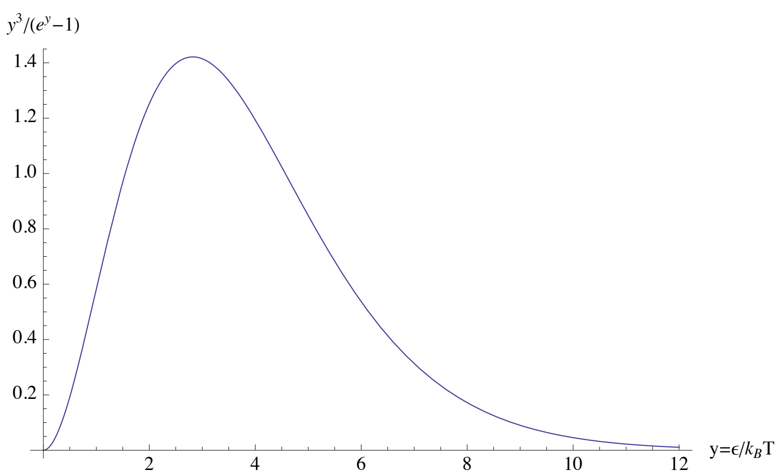
$$\lambda = \frac{2\,L}{n} \text{ and } p = \epsilon = \frac{h}{p} = \frac{h\,n}{2\,L}, \text{ where } n \text{ labels the mode.}$$

Changing the variable of integration from n to the dimensionless variable $x = \epsilon/K_B\,T = h\,n/2\,L\,k_B\,T$ and evaluating the integral over all space and all modes gives the spatial radiation energy density, ie, the relative intensity of the radiation per unit volume as a function of the photon energy (or frequency, if you change variables again to $f = \epsilon\,/h$), as,

$$\epsilon_{\text{rad}} = \frac{8\,\pi(k_B\,T)^4}{h^3}\int_0^\infty \frac{y^3}{e^y - 1}\,dy.$$

The integrand is the famous ***black-body spectrum***:

```
Plot[{y³/(Exp[y] - 1)}, {y, 0, 12}, AxesLabel → {"y=ϵ/k_BT", "y³/(e^y-1)"}]
```



The area under any segment of this curve from y to $y + dy$, when multiplied by $\frac{8\,\pi(k_B\,T)^4}{h^3}$, gives the energy density within the corresponding frequency range. Since this distribution has a strong peak at $\epsilon = 2.82\,k_B\,T$, we can take the average photon energy to be about $\epsilon_{\text{mean}} \cong 3\,k_B\,T$ or $f \cong 3\,k_B\,T/h$.

Actually performing the integral (see my notes on the derivation of Liddle eqn. (2.10)) and, restoring the c factor, gives a result of

$$\epsilon_{\text{rad}} = \frac{\pi^2\,k_B{}^4}{15\,\hbar^3\,c^3}\,T^4 = 7.565\times10^{-16}\,T^4\,J\cdot m^{-3}.$$

1.7.3 Neutrinos

Neutrinos are electrically neutral and therefore do not participate in electromagnetic interactions. They are ***leptons*** (particles with half-integer spin) and therefore not susceptible to the strong force. They interact only through the weak force and gravity. Since the weak force is extremely short-range and their mass is so tiny, they are barely detectable.

Liddle entertains the possibility that neutrinos are massless, but experimentally observed neutrino oscillations since the book was published require them to have a nonzero mass. The sum of the masses of all 3 types is expected to be less than 0.32 ± 0.081 eV, a mass less than one millionth that of the electron. Cosmology predicts a fixed ratio between the numbers of neutrinos and protons.

WMAP has found evidence for the existence of a cosmic neutrino background, similar to the photons of the CMB. Neutrinos made up a much larger part of the mass of the early universe than they do today — perhaps as much as 10% at the time the CMB was formed, 350,000 years post Big Bang. Today they constitute less than 1% of universal mass.

1.7.4 Dark matter

A small portion of dark matter may be composed of baryons — so called MACHOs, MAssive Compact Halo Objects, but a study of neucleosynthesis (Liddle, Chapter 12) requires the vast majority of dark matter to be non-baryonic. Many possibilities have been proposed, but the makeup of dark matter is still unknown.

1.7.5 Dark Energy

By far, the bulk of the mass-energy of the universe is dark energy. There is much speculation about its nature, but little is known for certain at this point.

1.8 The Density Parameter, Ω

Each of the constituents notes above contributes to the mass-energy density of the universe. As we noted in the section on curvature, it the ratio of the sum of all these densities to the critical density, ρ_c, that determines the overall curvature of space-time. The ratio of the density contribution of a particular component to ρ_c gives a measure of the relative importance of its contribution to the flatness of the universe that we observe. We call this ratio the ***density parameter***, $\Omega = \rho/\rho_c$.

The current density parameters are

Component	Symbol	□	Ω
Baryonic matter	$\Omega_{0,\text{bar}}$	□	0.049
Dark matter	$\Omega_{0,\text{DM}}$	□	0.268
Photons + neutrinos	$\Omega_{0,\text{rad}}$	□	8.24×10^{-5}
Dark energy	Ω_Λ	□	0.683

Because the various densities change over time (except possibly ρ_Λ) and the critical density, ρ_c, also changes, the various Ω's are also functions of time.

1.8 The Cosmological Constant, Λ

Because all matter-energy is gravitationally attractive, there are no solutions in GR that represent a static, homogeneous universe. What is needed is something that exerts an outward pressure to keep the whole thing from collapsing. We will see that introducing a "***cosmological constant***," Λ, that may or may not actually be constant, can offset this collapse and lead to the accelerating expansion that we see today. As in the table above, we can define a density parameter associated with Λ, called $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} \frac{\Lambda}{3 H^2}$. Note that, though Λ may be a constant, Ω_Λ is not. A universe in which $\Omega_\Lambda = 1$ is known as a ***deSitter universe*** and is the ultimate fate of our universe.

The mass density, ρ, is what we usually think of as a density - the amount of a substance contained in a unit volume. But, since Λ is a quality of spacetime and not a substance, ρ_Λ is somewhat more difficult to visualize. It is helpful to think of Λ is as a fluid that fills spacetime and has an energy density ρ_Λ and exerts a pressure, $p_\Lambda = -\rho_\Lambda c^2$. Since ρ_Λ is a positive constant, the cosmological constant exerts a constant outward pressure. It is this pressure that is the source of the current accelerating expansion of the universe.

2 General Relativity

Liddle, for the most part, avoids GR, relegating it to an "advanced topic" chapter and only covering it there with a broad brush. I think that, since cosmology has its roots firmly in the soil of GR, it's a good idea to dig a bit deeper. The following is still at a pretty descriptive level, but may help to establish a few important concepts.

2.1 The Metric Tensor

We are so accustomed to flat space that we automatically think in those terms, where the separation between objects, ds, is given by the ***Euclidean metric***, $ds^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates. In the notation of relativity, this is written as $ds^2 = \delta_{ij}\, dx^i\, dx^j$. This simple-looking expression includes four important GR conventions/constructs:

1) Roman letter indices indicate 1, 2, and 3. (Greek letter indices are used to stand for 0, 1, 2, and 3.)

2) The common spatial axis designations, x, y, and z, are replaced by x^1, x^2, and x^3, summarized as x^i or x^j. Note that i and j are indices, not powers. (When Greek indices are used this correspondence remains, and additionally, x^0 stands for time, t, or more correctly, ct.)

3) ***Einstein summation*** applies, in which a sum over repeated indices is assumed. Since i and j appear twice each on the right hand side, a sum over them as they run from 1 to 3 is implied. So, writing the metric in its full expression, $ds^2 = \sum_{i=1}^3 \sum_{j=1}^3 \delta_{ij}\, dx^i\, dx^j$. (The same summation convention applies to Greek indices, except that they run from 0 to 3.)

4) δ is the Kronecker delta; $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. In this application, δ_{ij} can be thought of as a rank-2 (2 indices) ***metric tensor***, a 3×3 matrix indexed by i and j that specifies coefficients for each of the nine products in the above sums. Generally the metric tensor is much more complicated, but from the previous definition of δ_{ij} we can see that in this case it's just the identity matrix:

$$[\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In special relativity, (SR), we deal with frames of reference that are stationary or moving at a constant velocity with respect to each other. These are called inertial frames or ***Lorentz frames***. Analysis of these reference frames shows that our intuitive concept of separate time and space coordinates leads to wrong conclusions at high relative velocities and we must "weld" them together into the a "***spacetime*** continuum." Hence the need for Greek indices that run over the 4 dimensions of space and time.

Now we write the above metric as $ds^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta$, using Greek indices α and β to indicate 4 values for the 4 dimensions. Here $\eta_{\alpha\beta}$ is the ***Minkowski metric tensor*** for flat spacetime. It too is simple:

$$[\eta_{\alpha\beta}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note though, that the time component now has a different sign from the spatial ones, so the Einstein summation could result in a negative value for ds^2. (The Euclidean metric is always non-negative.)

[Note also that this definition and some of the following equations differ from those in Liddle's book by a minus sign. This difference reflects my preference for the so called "mostly minus metric" in contrast to Liddle's use of the "mostly plus metric." Thus, Liddle would write,

$$[\eta_{\alpha\beta}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Unfortunately, there is not a single convention for this and it's one of those annoying complications in physics that just make already difficult concepts a little bit more confusing. Sorry about that, but I'm writing this for me, so I get to do it the way I'm most comfortable!]

In SR, spacetime is always flat — the Minkowski metric always applies. In GR, though, we no longer have the guarantee of the comforting normalcy of flat spacetime. Not surprisingly, this means that the metric tensor can be much more complicated. Now we write the metric as, $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$. (We could still use α and β, but μ and ν are conventional.) This new metric tensor, $g_{\mu\nu}$, now stands for any general metric tensor, even ones that apply to curved spacetime. The cosmological principle, though, allows us to limit the ones that concern us by requiring that, neglecting local variations, spacetime must have the *same* curvature everywhere.

The simplest of these cases is the ***Friedmann-Lemaître-Robertson-Walker (FLRW) metric tensor*** for an expanding, flat spacetime. As you can probably guess, this is just,

$$g_{\mu\nu}^{\text{FLRW}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}.$$

So far, we have worked in Cartesian coordinates, but, since we are looking radially outward from earth, spherical spatial coordinates (r, θ, ϕ) are more natural. If we incorporate these and construct a general metric for a curved, expanding spacetime, we get the ***Robertson-Walker (RW) metric***,

$$ds_{\text{RW}}{}^2 = c^2\, dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\,\theta^2 + \sin^2\theta\, d\,\phi^2\right)\right].$$

The spacetime described by this metric is the most general one that is both homogeneous and isotropic. The time coordinate, called ***cosmic time***, is the time of a ***fundamental observer*** whose only motion is due to the expansion or contraction of spacetime. The spatial coordinates (r, θ and ϕ) assigned by a fundamental observer are comoving coordinates and, at any specific cosmic time, all fundamental observers will be measuring the same 3-dimensional ***space-like hypersurface*** embedded in 4-dimensional spacetime. Each such hypersurface represents all of space at that moment of cosmic time. It can be thought of in a 3-dimensional analogy as the 2-dimensional surface that is formed by the events corresponding to the same instant in cosmic time on the diverging world lines of all fundamental observers in expanding spacetime.

We would like to know how such a spacetime evolves, since that will contain important clues to the evolution of our own universe. This evolution is specified by the ***field equations*** of GR. So, let's take a brief detour to understand those field equations.

2.2 Curvature

We start with some definitions. We call a smoothly curved space that is everywhere locally flat, a ***manifold***. The surface of a sphere, for example is a manifold — small parts of it look flat. The surface of a cone is not a manifold, since, no matter how closely you look, its apex is not flat. The curved spacetime of GR is a manifold, but it's a manifold with two additional properties:

1) It's ***differentiable***. Almost all spaces that concern physicists are differentiable, since that provides some vitally important characteristics and seems, thankfully, to be the way nature operates.
2) Its metric tensor must be ***symmetric*** (ie, $g_{\mu\nu} = g_{\nu\mu}$).

Such manifolds are called ***Riemann manifolds***. They were first studied in depth by Bernhard Riemann in the mid 19th-century. (Actually, the manifolds of GR are ***pseudo-Riemann manifolds***, since they can have, as noted above, a positive, zero or negative ds^2.) Riemann was able to show that he could capture all curvature information about these manifolds in a rank-4 tensor called the ***Riemann curvature tensor***, $R^{\alpha}_{\beta\gamma\delta}$. $R^{\alpha}_{\beta\gamma\delta} = 0$ implies a flat manifold.

GR doesn't actually use the full $R^{\alpha}_{\beta\gamma\delta}$, but instead uses two other measures of curvature derived from it. A tensor can have its rank lowered by a process called ***contraction*** that amounts to multiplying it by another tensor with the same index in the opposite position (up or down). [Think of turning a vector (a rank-1 tensor) into a scalar (a rank-0 tensor) by forming a dot product with another vector.] When the first and last indices of the Riemann curvature tensor are contracted, the result is the rank-2 ***Ricci tensor***, $R_{\mu\nu}$. Another contraction over the remaining two indices results in the ***Ricci scalar***, $R = g^{ij} R_{ij}$. Einstein then collected these two entities into one — the ***Einstein tensor***, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$.

The ***Einstein field equations*** relate the curvature of spacetime to the configuration of mass-energy that causes the curvature. We have seen that curvature is captured in the Einstein tensor, but we need a way to describe the energy and momentum of the collection of particles whose gravity is the source of the curvature. The answer is $T^{\mu\nu}$, the symmetric, rank-2 ***energy-momentum tensor*** (sometimes called the "stress-energy tensor"). Consider a collection of particles moving along their individual world lines. Each carries its four-momentum (a combination of energy and momentum) with it, and the collection forms a "river of four-momentum" flowing through spacetime. The density and flow of this river is the source of the general relativistic gravitational field and is captured in $T^{\mu\nu}$.

The Einstein field equations then relate $G_{\mu\nu}$ to $T_{\mu\nu}$ in the famous expression, $G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$. This is often written in "geometrized" units, where $8\pi G = c = 1$, as the frustratingly obscure $G_{\mu\nu} = -T_{\mu\nu}$. This equation relates two symmetric 4×4 matrices, thus representing, at most, 10 independent equations (symmetry requirements eliminate 6 of the 4×4=16 equations). There are, however, two important special cases that, because of their symmetries, significantly reduce this number. These are:

1) ***Dust***: Dust is a collection of non-interacting particles at rest with respect to each other in a Lorentz frame. Dust is a reasonable approximation to the late stages of a matter-dominated universe. Since they have no momentum, $\gamma = 1$ for the dust particles and, the energy-momentum tensor is the particularly simple expression,

$$\left[T^{\mu\nu}_{\text{Dust}}\right] = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$, where mass density $\rho = n\,m$, with n being the number density of the particles.

The field equations for dust become a single equation.

2) ***Perfect fluids***: A perfect fluid is best thought of as dust with a pressure that acts with equal magnitude in all directions. It's a good model for the universe at times earlier than the ones modeled by dust alone. Prefect fluids have zero viscosity and zero heat conductance. No viscosity implies that the fluid cannot support any sheer stress, so all off-diagonal elements of its $T_{\mu\nu}$ are zero. Thus, T_{ij} must be a diagonal matrix. Because of the cosmological principle, it must be diagonal *in all frames*, so must be a scalar multiple of the 3×3 identity matrix. Because pressure is the result of a force directed perpendicular to the interface between particles, $T_{ij} = \delta_{ij}\,p$, where p is now pressure (not momentum!). (In the case of a universe dominated by non-exotic matter, where $p \ll \rho$, the perfect fluid case reduces to the simple case of dust.) Thus,

$$\left[T^{\mu\nu}_{\text{Perfect Fluid}}\right] = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

The field equations for a perfect fluid then become two equations, a scalar one in time and a 3-vector one for the spatial components. Starting with the Robertson-Walker metric and passing through some intermediate steps (involving the covariant derivative, Christoffel symbols, the Riemann curvature tensor and the Ricci tensor and scalar) which Liddle glosses over and I will too, it is possible to arrive at the ***Friedmann equations***,

$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k\,c^2}{a^2}$, (what Liddle calls "*the* Friedmann equation") and

$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho - \frac{3\,p}{c^2}\right)$, (what Liddle calls "the acceleration equation").

These can be combined to derive the, non-independent, "fluid equation,"

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\left(\rho - \frac{p}{c^2}\right) = 0.$$

3 Cosmological Models Without Λ

The Friedmann equation is the key to understanding how the curvature, k, determines the geometry, and hence the behavior, of the universe. We are interested in how the scale factor, $a(t)$, behaves under these different scenarios. As Liddle points out, k is often set to one of only three values, ±1 or 0. The disadvantage of this approach is that we can no longer set $a_0 = 1$. This is a serious limitation. In this section, though, I will set k to only those values because my purpose is to explore qualitatively the behavior of open, closed and flat universes rather than to draw any particularly useful quantitative conclusions.

For this we will need the ***deceleration parameter***, q, introduced in section 6.3 of Liddle. Liddle discusses q only at the present time, so limits his coverage to the constant q_0. The deceleration parameter, though, can be more generally thought of as a function of time,

$$q(t) \equiv -\frac{\ddot{a}(t)}{H^2(t)\,a(t)} = \frac{\rho(t)}{2\,\rho_c(t)},$$

and we will be interested in its behavior given various conditions. Cosmologists have recently learned that the universe is currently expanding at an accelerating rate, so $q_0 < 0$. Such an accelerating expansion cannot be accommodated here, since the universes considered in this section are without a ***cosmological constant***, Λ.

3.1 Characteristics of Flat Universes

We first examine the properties of flat matter-dominated and radiation-dominated universes without a cosmological constant (dark energy), and then tabulate their behavior for other curvatures.

3.1.1 The Flat Non-relativistic Matter-dominated Universe

These are universes that are modeled on the pressureless dust approximation. This gives an acceleration equation that reads,

$$\frac{\ddot{a}}{a} + \frac{4\pi G}{3}\rho = 0, \text{ or } \rho(t) = \frac{3H^2(t)}{4\pi G}q(t).$$

The Friedmann equation,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

then becomes,

$$H^2 - 2H^2 q = -\frac{k}{a^2}, \text{ or } k = a^2 H^2 (2q - 1).$$

Taking $k = 0$, ($a > 0$ and $H > 0$) this gives $q = 1/2$ and, because at $k = 0$, $\rho = \rho_c$ the critical density is,

$$\boxed{\rho_c(t) = \frac{3H^2}{8\pi G}}.$$

The fluid equation,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0$$

can be rearranged to give,

$$a^3\dot{\rho} + 3\rho\dot{a}a^2 = 0.$$

The clever mathematicians among us will recognize this as a product rule differentiation,

$$\frac{d}{dt}\left(\rho a^3\right) = \dot{\rho}a^3 + 3\rho\dot{a}a^2 = 0 \text{ which implies } \rho \propto a^{-3}, \text{ or, with initial conditions, } \boxed{\rho(t) = \rho_0\left(\frac{a_0}{a(t)}\right)^3 \propto a^{-3}}.$$

Using this in the Friedmann equation, with $a_0 = 1$, gives,

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3a}.$$

Liddle solves this with the clever power law guess that $a \propto t^q$, but this separable differential equation also easily yields to integration as,

$$\dot{a} \propto a^{-1/2} \to a^{1/2}\,da \propto dt \to \int a^{1/2}\,da \propto \int dt \to a^{3/2} \propto t \to \boxed{a \propto t^{2/3}}.$$

Substituting into the above expression for ρ gives, $\boxed{\rho_{\text{matter}} \propto t^{-2}}$.

The conformal time is given by,

$$\eta \equiv \int_0^t \frac{dt'}{a(t')} = \int_0^t \left(\frac{t_0}{t'}\right)^{2/3} dt' \propto t^{1/3} \to \boxed{\eta \propto a^{1/2}}$$

3.1.2 The Flat Radiation-dominated Universe

These universes are modeled with the perfect fluid, $p = \frac{\rho}{3}$, approximation. Virtually identical analysis to the above gives,

$$\boxed{\rho(t) = \rho_0\left(\frac{a_0}{a(t)}\right)^4 \propto a^{-4}};\ \boxed{a \propto t^{1/2}};\ \boxed{\rho_{\text{rad}} \propto t^{-2}} \text{ and } \boxed{\eta \propto a}.$$

It should be noted that the rate at which density decreases as a function of a is faster for a radiation-dominated universe than for a matter-dominated universe and therefore, regardless of how small a component of matter is present in the early universe, it will eventually come to dominate. Radiation-dominance is an unstable condition. As a universe transitions from radiation-dominance to matter-dominance, the expansion rate will speed up from $\dot{a} \propto t^{-1/2}$ to $\dot{a} \propto t^{-1/3}$.

■ 3.2 Specific Solutions Based on Curvature

The following tables give the evolution of the scale factor for matter- and radiation-dominated universes for each of the three canonical curvatures, $k = 0, \pm 1$. Because our universe appears to be flat to a high degree of certainty and dominated by a cosmological constant, none of these solutions is of more than theoretical interest, so I will just tabulate the results rather than derive them.

Matter-dominated Universes

Characteristics		for all η		for small η		
k	q_0	a	t	$a(\eta)$	t	$a(t)$
0	$= 1/2$	$(6 \pi G \rho_0)^{1/3} t^{2/3}$	–	–	–	$\propto t^{2/3}$
+1	$> 1/2$	$\left(\frac{q_0}{2 q_0 - 1}\right)(1 - \cos \eta)$	$\left(\frac{q_0}{2 q_0 - 1}\right)(\eta - \sin \eta)$	$\propto \eta^2$	$\propto \eta^3$	$\propto t^{2/3}$
−1	$< 1/2$	$\left(\frac{q_0}{1 - 2 q_0}\right)(\cosh \eta - 1)$	$\left(\frac{q_0}{1 - 2 q_0}\right)(\sinh \eta - \eta)$	$\propto \eta^2$	$\propto \eta^3$	$\propto t^{2/3}$

Radiation-dominated Universes

Characteristics		for all η		for small η		
k	q_0	a	t	$a(\eta)$	t	$a(t)$
0	$= 1/2$	$\left(\frac{32 \pi G \rho_0}{3}\right)^{1/4} t^{1/2}$	–	–	–	$\propto t^{1/2}$
+1	$> 1/2$	$\sqrt{\frac{2 q_0}{2 q_0 - 1}}\ (\sin \eta)$	$\sqrt{\frac{2 q_0}{2 q_0 - 1}}\ (1 - \cos \eta)$	$\propto \eta$	$\propto \eta^2$	$\propto t^{1/2}$
−1	$< 1/2$	$\sqrt{\frac{2 q_0}{1 - 2 q_0}}\ (\sinh \eta)$	$\sqrt{\frac{2 q_0}{1 - 2 q_0}}\ (\cosh \eta - \eta)$	$\propto \eta$	$\propto \eta^2$	$\propto t^{1/2}$

■ 3.3 Particle Number Density

We have been concerned so far with mass-energy density, ρ, but another useful density, ***number density***, n, simply counts the *number* of particles of a particular type in a unit volume. It's useful because a system at thermal equilibrium will statistically preserve the number of each type of particle, since, by definition, equilibrium reactions that produce or destroy particles run at the same rate in both directions. Number density will change in an expanding universe, though, because expansion changes the volume, so, $n \propto a^{-3}$.

This relation is exactly what we would expect for non-relativistic matter ($\rho_{\text{matter}} \propto a^{-3}$), but we have seen that radiation energy density falls off more quickly ($\rho_{\text{rad}} \propto a^{-4}$). The extra factor of a^{-1} is due to the loss of energy by the photons as their wavelengths are stretched in the expansion. So, even though energy densities of matter and radiation evolve differently in an expanding universe, their number densities evolve in exactly the same way, $n \propto a^{-3}$.

4 Cosmological Models With Λ

■ 4.1 The Cosmological Constant

Einstein quickly realized that, if the energy sources in the energy-momentum tensor are only matter ($p_{\text{mat}} = 0$) and radiation $\left(p_{\text{rad}} = \rho_{\text{rad}}\, c^2 / 3\right)$, there are no solutions to the field equations of GR that describe a static, homogeneous universe. To remedy this situation and bring GR into alignment with the astronomical thinking of the day, he proposed including a "cosmological constant," Λ, ($p_\Lambda = -\rho_\Lambda$) so that his field equations now read $R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$. The Friedmann equation that results from this includes another term that incorporates Λ,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k c^2}{a^2} + \frac{\Lambda}{3}.$$

Such an equation admits a static ($H(t) = 0$), homogeneous solution with positive matter density and curvature. Unfortunately, it wasn't long after he proposed this idea that Einstein realized that such a universe was also unstable (See Liddle problem 7.2).

The acceleration equation makes the effect of the inclusion of Λ explicit,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho - \frac{3p}{c^2}\right) + \frac{\Lambda}{3}.$$

Thus, a positive Λ makes a positive contribution to $\ddot{a}$, effectively creating a repulsive force. If $\Lambda > 4\pi G\left(\rho - \frac{3p}{c^2}\right)$, it results in an accelerating expansion.

■ 4.2 Characteristics of Universes with $\Lambda \neq 0$

The inclusion of Λ changes the simple conclusions about the behavior of universes based only on their curvature. A cosmological constant allows for the possibility that a closed universe may not collapse or that an open one expands forever. The various possibilities are best parameterized using a graph of Ω_Λ versus Ω_0 shown in Liddle as Figures 7.1 and A2.4. For an accelerating, flat, pressureless universe with a cosmological constant, $\Omega_\Lambda > 1/3$. In a universe like ours, with $\Omega_0 \leq 1$, a future re-collapse depends on the sign of Λ; $\Lambda \geq 1$ implies endless expansion.

Our universe has had/will have 3 different epochs in its history:

radiation domination (0 - 10^4yrs),

matter domination (10^4 - 10^{10}yrs), and

dark energy domination (> 10^{10}yrs).

During the first two, expansion was at a sub-linear rate (slowing rate of expansion), that becomes exponential during the third.

■ 4.3 The Mysteries of Λ

■ 4.3.1 Zero-point Energy of the Vacuum

Today, we see the cosmological constant, Einstein's "greatest blunder," as a masterful stroke of genius that allows us to both accommodate a required non-zero vacuum energy and explain the observed accelerating expansion of the universe. If the vacuum is to be Lorentz invariant, its energy-momentum tensor must have the form $T_{\mu\nu} = -\rho_\Lambda\, g_{\mu\nu}$ and must be formally equivalent to Λ, with $\rho_\Lambda = \frac{\Lambda}{8\pi G}$. It is this equivalence that leads to the association between the cosmological constant and the vacuum energy.

Quantum field theory requires that the underlying reality is composed of a set of quantum fields whose fixed frequency modes each behave like simple harmonic oscillators whose ground state energy is $E_0 = \frac{\hbar\omega}{2}$. In a situation reminiscent of the ultraviolet catastrophe, the integral of the energy over all of these modes diverges. In a similar (but much more *ad hoc*) solution, QFT resolves this paradox by limiting the contribution of modes with a wavelength smaller than some Planck scale cutoff. Such a calculation leads to a vacuum energy density of $\rho_\Lambda \cong 10^{112}\,\text{erg}\cdot\text{cm}^{-3}$. Measurements of Type Ia supernovae and CMB anisotropies leads to a value of $\rho_\Lambda \cong 10^{-8}\,\text{erg}\cdot\text{cm}^{-3}$, resulting in the infamous discrepancy of a factor of 10^{120} between theory and experiment. It remains a mystery why the measured ρ_Λ is so wildly different from the QFT prediction and why it has an exceedingly small, but non-zero, value.

4.3.2 Why are Ω_Λ and Ω_{mat} of Roughly the Same Size?

If we lump dark matter with ordinary baryonic matter, the matter and dark energy density parameters are, $\Omega_{\text{mat}} \cong 0.3$ and $\Omega_\Lambda \cong 0.7$. Since $\rho_{\text{mat}} \propto a^{-3}(t)$ and $\rho_\Lambda \propto a^0(t)$, this near equality of density parameters exists for a very small portion of the history of the universe. It appears that the transition from a matter-dominated to a cosmological constant-dominated universe, with its inherent accelerated expansion, is a relatively recent phenomenon. Why we happen to live during this particular period in the history of the universe also seems to demand an explanation. There is no consensus on an answer.

4.4 Some Possible Solutions

Over the last decade and a half, almost everyone in the field of cosmology has come to accept the model of a universe expanding at an accelerating rate with $\Omega_{\text{mat}} \cong 0.3$ and $\Omega_\Lambda \cong 0.7$. There are a few holdouts who continue to support a model of unaccelerated expansion with $\Omega_{\text{mat}} \cong 1$, but these theories, all involving various forms of invalidation of high-redshift observational data, seem increasingly contrived and untenable. Assuming that the accelerated expansion is real, there are several possible ways out of the conundrum.

4.4.1 Failure of GR

If we accept that we live in an accelerating, isotropic and homogeneous universe, GR in its current form is unambiguously clear that some source of negative pressure (“dark energy” for lack of a better term) is required. A dark energy-free acceleration can arise and the observed universe can be postdicted by requiring a modification of GR at cosmic scales.

Some string theorists have proposed ideas of this sort that require modifications of the Friedmann equation. I don’t understand these ideas very well, but they seem to require gravity to be four-dimensional below a certain (very large) length scale and higher-dimensional above it.

Others have proposed four-dimensional modifications to GR at all scales. In an elegant approach similar to Feynman’s path integral formulation of QM, the Einstein field equations can be derived by minimizing an action given by the spacetime integral of the Ricci curvature scalar, R,

$$S = \int d^4 x \sqrt{g_{\mu\nu}}\; R.$$

Theorists proposing a modification to GR that does not require the additional dimensions of string theory, simply add a $1/R$-dependent term to the integrand,

$$S = \int d^4 x \sqrt{g_{\mu\nu}} \left(R - \frac{\mu^4}{R}\right), \text{ where } \mu \text{ is a tunable parameter.}$$

These theories admit accelerating solutions, but make other predictions that appear to conflict with observations.

Unfortunately, all of these approaches to modifying GR lead to some fine-tuning mysteries of their own and, to my untrained eye, seem to add complexity (and probably inconsistencies with observation) without really solving the underlying problems inherent in the accelerating expansion.

4.4.2 Varying Λ

The Friedmann equation with Λ requires that $\dot{a}^2 \propto a^2 \rho + \textit{constant}$. So, for acceleration to occur, the dark energy density must fall off more slowly than a^{-2}. Neither matter ($\rho_{\text{mat}} \propto a^{-3}$) nor radiation ($\rho_{\text{rad}} \propto a^{-4}$) does the job. A constant Λ ($\rho_\Lambda = \text{constant}$) will work, but so will a slowly decreasing Λ. It offers a potential solution to the problem of a small but non-zero cosmological constant: Λ has been decreasing for a long time, falling asymptotically toward zero, and we happen to live at the time when it has its current tiny value. It is possible that the problem of the close values of Ω_{mat} and Ω_Λ might also be resolved in this way.

The simplest of these possibilities is a scalar inflaton field dropping slowly in a weak potential field. (If the potential were stronger, Λ would presumably have already reached its minimum.) This can be thought of as a "quintessence fluid" with an equation of state given by, $p_Q = w\,\rho_Q\,c^2$, where w is a constant. The $w = -1$ case is the familiar cosmological constant, with expansion occurring for $w < -1/3$. This approach is known as ***quintessence***. In QFT, when weak (low mass) scalar fields are subject to renormalization, it drives their masses up, so fields as weak as those required for quintessence are unnatural in QFT and so quintessence introduces other fine-tuning problems.

There are other theories that posit an oscillating ρ_Λ superimposed on an exponential decay. This could help with the unlikelyhood of finding ourselves so near the transition to dark energy-dominance. Such transitions happens fairly frequently - once per cycle.

All these theories have issues, either with fine-tuning of parameters or constraints due to neucleosynthesis. Additionally, none of them offer a good motivation for a varying Λ.

■ 4.4.3 Supersymmetry

Supersymmetry offers some intriguing hints about the size of the vacuum energy. For every bosonic degree of freedom (contributing a positive vacuum energy) there must be a partner fermionic degree of freedom (contributing a negative vacuum energy). If the degrees of freedom match, vacuum energy would be zero. However it is clear that the world in which we live is not in a supersymmetric state - there is no bosonic selectron with the same mass as the electron, for example. So, supersymmetry, if it exists, must be a broken symmetry. The good news is that supersymmetry renders the vacuum energy finite and calculable (at least with some assumptions about the symmetry-breaking energy). The bad news is that such calculations give a result that is far outside observational limits. Subtleties in string theory and supergravity allow for tuning the result, but, once again are entirely *ad hoc*.

■ 4.4.4 Anthropocentricity

The simplest answer to the mysteries of Λ is, of course, that we find ourselves in the only sort of environment, and at the only time in its evolution, that can accommodate human life. The vacuum energy, then, has a value that is an arbitrary feature of the region in which we find ourselves. Such regions must be at least as large as the observable universe and there must be some mechanism by which Λ can vary between them. String theorists propose a string theory "landscape" in which fundamental "constants" such as Λ can take on different values in different universes or parts of this universe. Inflation offers a possible mechanism for this, by which different parts of the multiverse undergo rapid inflation with different resulting properties.

5 The Evolution of the Universe

■ 5.1 The Age of the Universe

In a flat universe with a positive cosmological constant and $\Omega_0 < 1$, its calculated age is greater than our initial, naive estimate of the Hubble time, $H_0^{-1} = 9.77\,h^{-1} \times 10^9$yrs. The expression for this, more accurate age is,

$$t_0 = H_0^{-1}\left[\frac{2}{3}\,\frac{1}{\sqrt{1-\Omega_0}}\,\ln\left(\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right)\right] = H_0^{-1}\left[\frac{2}{3}\,\frac{1}{\sqrt{1-\Omega_0}}\,\sinh^{-1}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right)\right].$$

With measured values for Ω_0 and h, this calculation gives an age of $13.7 \pm 0.1 \times 10^9$ years.

There are several sources of observational data that support such an estimate:

Uranium isotopic ratios in the galactic disk,
the cooling rate of white dwarf stars, and
the chemical evolution of stars in globular clusters.

All give ages in rough agreement with the calculation.

5.2 The Composition of the Universe, $\Omega = \Omega_0 + \Omega_\Lambda$

The total matter-energy of the universe includes ordinary (baryonic) matter, most of which is dark; non-baryonic matter, all of which is dark; and dark energy. Thus, almost all of the mass-energy density of the universe is hidden from us. We know it exists, though, from several lines of investigation that do not involve measuring its emission or absorption of EM radiation.

5.2.1 The Mass Density Component of the Universe, Ω_0 (31.7%)

The mass density of the universe, Ω_0, is composed of baryonic matter - the stuff with which we are familiar, stars, gas clouds and other atomic and sub-atomic matter - and non-baryonic matter - matter that carries no electric charge and does not interact electromagnetically. Three lines of evidence suggest that the majority of universal mass is non-baryonic:

1) ***Big bang neucleosynthesis***, which accounts for the relative abundances of elements in the early universe, predicts that baryonic matter cannot account for more than about 5% of the mass-energy density of the universe. This is because non-baryonic matter does not contribute to the formation of elements in the early universe.

2) Studies of ***gravitational microlensing*** have shown that only a small fraction of the dark matter in our galaxy (and presumably in other galaxies as well) can be in large dark objects like burned out stars.

3) The geometry of the universe has been measured with great accuracy by examining the structure size in the ***microwave background radiation*** and has been found to be flat to an accuracy of less than 1%. This, in turn places stringent requirements on the total mass density of the universe resulting in $\Omega_0 \cong 31.7\ \%$.

Galaxy rotation curves that compare a star's rotational velocity to its radial distance show typical velocities of stars at large radii to be as much as three times what they should be if all galactic matter were luminous. This implies that the galaxy consists of considerably more dark mass than that which is luminous. In fact, the amount of this dark matter necessary to account for rotation curves considerably exceeds that allowed by nucleosynthesis, so most of it must be non-baryonic.

5.2.1.1 Baryonic Matter, $\Omega_B \cong 4.9\ \%$

The theory of nucleosynthesis places limits on the amount of baryonic matter that the universe can contain. In fact, to match the observed element abundances, the amount of baryonic matter must exhibit a mass density parameter in the range $3.1\ \% \leq \Omega_B \leq 5.0\ \%$. This baryonic matter is further divided between that which glows in the electromagnetic spectrum (luminous matter), Ω_{BL}, and that which does not (baryonic dark matter) Ω_{BD}.

5.2.1.1.1 Luminous Matter, $\Omega_{BL} < 1\ \%$

All luminous matter is baryonic. Estimates of the amount of luminous matter in the universe begin with that contained in stars. Estimates of the stellar density parameter are in the range $0.5\% < \Omega_{stars} < 1\%$. Since this is less than the range required for baryonic matter by the theory of nucleosynthesis, there must be considerable baryonic matter not in stars. Additionally, there exists interstellar gas, some of which glows in the X-ray spectrum, but this luminous gas does not represent enough matter to cause Ω_{BL} to exceed about 1%.

5.2.1.1.2 Baryonic Dark Matter, $\Omega_{BD} \cong 4\ \%$

We saw above that some intergalactic gas shines in X-rays due to the gravitational effect of galaxy clusters. Since only a small fraction of galaxies are in clusters, it seems a reasonable assumption that there exists a large body of cool gas between galaxies that is dark and which contributes to the baryonic dark matter total. We also expect that some baryonic matter resides in very low mass stars(white or brown dwarfs) or large planets that do not emit light or radiate so faintly that we cannot detect them. One category of these are the MAssive Compact Halo Objects (MACHOs).

5.2.1.2 Non-baryonic Dark Matter, $\Omega_{NB} \cong 26.8\ \%$

Baryonic matter cannot provide sufficient mass to create the necessary gravitational attraction to explain galaxy rotation curves, so dark matter must be more than simply dark baryonic matter. The only currently known particle whose properties are sufficiently uncertain that it could contribute to dark matter is the neutrino. However, studies of large-scale structure and high redshift galaxies lead to the conclusion that neutrinos can contribute only a small fraction (probably < 2%) of the necessary non-baryonic mass.

Theories of non-baryonic matter are classified by the ***free-streaming length*** of the particle(s) involved. Free-streaming length refers to the distance the particles could have moved in the early universe before being slowed by inflation and it sets a minimum scale for structure formation. There are three current hypotheses for non-baryonic dark matter: cold dark matter (CDM), warm dark matter (WDM) and hot dark matter (HDM). Some combination of these is also possible.

5.2.1.2.1 HDM

Hot dark matter particles are light particles that remain relativistic until shortly before recombination. The best candidate for an HDM particle is the neutrino, but as we have seen, it falls far short of accounting for the bulk of the mass necessary for non-baryonic dark matter.

5.2.1.2.2 WDM

Warm dark matter particles have a free-streaming length similar to the size of a proto-galaxy and a mass on the order or 1keV. They became non-relativistic when the universe was about 1 year old - during the epoch of radiation-domination. Supersymmetric gravitinos and photinos have been suggested as candidate particles, but these have not been seen in experiments to date. Also postulated is the ***sterile neutrino***, an unobserved massive particle that does not even interact by the weak force, as ordinary neutrinos do.

5.2.1.2.3 CDM

Cold dark matter particles are those that are massive enough to have become non-relativistic very early, and so diffused a very short distance. Candidates for CDM particles include Weakly Interacting Massive Particles (WIMPs) such as massive supersymmetric particles, or small black holes (also included in the MACHO category). It appears that MACHOs of any sort cannot account for the necessary non-baryonic mass density.

The leading candidates for the bulk of non-baryonic dark matter are the, as yet undiscovered, supersymmetric WIMPs.

5.2.2 The Dark Energy Component of the Universe, Ω_Λ (68.3%)

By far, the bulk of the mass-energy in the current universe is dark energy. There is much speculation about the nature of dark energy, but little is known for certain at this point. The two most commonly proposed forms are fixed quantities such as the cosmological constant, or scalar fields such as quintessence or modulii that vary over time and space. These can be distinguished by measurements of the evolution of the expansion rate of the universe but sufficiently precise measurements are not yet available.

5.3 The Standard Model of Cosmology

Combining the FLRW metric of GR with cold dark matter and the cosmological constant, Λ, results in what is known as the ***Lambda-CDM*** model of the universe. This is also often referred to as the ***Standard Model of Cosmology***. ΛCDM is often extended by including inflation.

6 Observational Cosmology

The goals of observational cosmology are to assess which of the theoretical models best fit the universe. In the following, we will assume that the Robertson-Walker metric applies.

6.1 Light Propagation and the Observable Universe

Redshift is the most fundamental result of observational cosmology. For light, the spatial part of the RW metric is equal and of opposite sign of the time part, so $ds = 0$ - light is motionless in spacetime. For a radial light ray ($d\theta = d\varphi = 0$), then the RW metric allows us to show that, for any spatial geometry,

$$\frac{a(t_0)}{a(t_e)} = 1 + z.$$

Astronomers frequently use redshift as a measure of time or distance, so the phrase "the universe at redshift z" means the universe at the time when it was $1/(1+z)$ of its present size, and "an object at redshift z" means one at a distance such that in the time it has taken its light to reach us, it has redshifted by a factor $(1+z)$.

If we assume a flat, matter-dominated universe with no cosmological constant, we can use the metric to calculate the finite size of the ***observable universe***, $r_0 = 3\,c\,t_0$. The assumption of no Λ is, of course, unrealistic, but, even worse, this result assumes a linear evolution of $a(t)$. A better definition is to equate the observable universe with that region from which we receive EM radiation, thus excluding spacetime events prior to the formation of the CMB, when the universe became transparent. It should also be noted that, because the universe is expanding as light travels, the distance light has traveled is greater than c times the age of the universe.

6.2 Luminosity Distance

Distant objects in a static universe are dimmer than nearby ones by the inverse square law for EM radiation. It is this dimming that we use when we calculate their ***luminosity distance***.

If an object has a ***luminosity*** of L units of energy emitted per steradian per second, its total power output is $4\pi L$. The radiation ***flux density***, S, is defined as the amount of energy received by us per unit area per second, and the luminosity distance, is given by

$$d_{\text{lum}} = \sqrt{\frac{L}{S}}\ .$$

This is not the actual distance to the object, though. It's too big because in an expanding universe,

1) the energy of the individual photons decreases proportional to $1/(1+z)$, and

2) there are less of them by a factor of $1/(1+z)$.

These two effects give a luminosity distance of $d_{\text{lum}} = a_0\, r_0 (1+z)$, while in a flat static universe their distance would be $d_{\text{phys}} = a_0\, r_0$. Of course, for nearby objects $z \ll 1$, so $d_{\text{lum}} \cong d_{\text{phys}}$, but for distant objects $d_{\text{lum}} > d_{\text{phys}}$. Non-flat geometries either enhance ($k < 0$) this trend or oppose ($k > 0$) it. Presence of Λ complicates the situation still further.

Luminosity distance, while very useful, has are several limitations.

The luminosity, L, we have used above is the integral of the power output of the radiator over all frequencies, called its ***bolometric luminosity.*** The luminosity of any radiator, though, is a (usually complicated) function of frequency. Due to redshift, the detector (sensitive to only a narrow band of wavelengths) is seeing light that was emitted in a different part of the spectrum of a distant object from that emitted by a closer object. That part of the spectrum may have a greater or smaller energy density than is radiated in the spectral band where it appears. This can be corrected for if we know the emission spectrum of the object, but as a practical matter this introduces uncertainty to distance measurements based on luminosity. Different cosmological models predict different relationships between luminosity distance and redshift.

In astronomical observations, the measured quantity is flux density and its translation into luminosity distance requires knowledge of the absolute luminosity of the object. We have only limited knowledge of this for distant (younger) objects. We can circumvent this problem by observing a class of objects at different distances thought to have the same absolute luminosity, even if we don't know what that luminosity is. This has been done using Type Ia supernovae, whose light curve is related in a known way to their relative luminosity. It was these measurements that lead to the recognition of an accelerating expansion and the need for a cosmological constant.

6.3 Angular Diameter Distance

As with luminosity distance, calculation of ***angular diameter distance*** assumes a flat, static universe. The angular measure of the extent of a distant object perpendicular to the line of sight is,

$$d_{\text{diam}} \equiv \frac{l}{\sin\theta} \cong \frac{l}{\theta}.$$

We again use the RW metric, this time with $r = r_0$, $d\,r = d\,\varphi = d\,t = 0$, and $t = t_e$, $l = d$, and $s = r_0\, a(t_e)\, d\,\theta$, with the scale factor accounting for any universal expansion during the travel time of the light. Thus,

$$d\,\theta = \frac{l\,(1+z)}{a_0\, r_0} \text{ and, } d_{\text{diam}} = \frac{a_0\, r_0}{(1+z)} = \frac{d_{\text{lum}}}{(1+z)^2}.$$

As with luminosity distance, the angular diameter distance of nearby objects closely matches their physical distance. However, the angular diameter distance behaves quite differently from the luminosity distance for distant objects. More distant objects, as expected, are fainter (d_{lum} increases) but, past $z = 5/4$ for a flat, matter-dominated universe, their angular size *increases* (d_{diam}decreases). Because the inverse square effect dims the light of distant galaxies and their angular diameter increases, their dimming light is spread over a larger area, so brightness decreases strongly with redshift.

Both angular diameter distance and luminosity distance are larger in universes with $\Lambda > 0$ and smaller in those with $\Lambda < 0$.

6.4 Source Counts

If we use the cosmological principle to assume that sources (usually galaxies of a chosen type) are uniformly distributed, with a number density $n(t) \propto a^{-3}$, and we use the volume element from the RW metric, we find that the total number of sources per steradian out to a distance r_0, known as the ***source count***, is,

$$N(r_0) = n(t_0)\, a_0{}^3 \int_0^{r_0} \frac{r^2}{\sqrt{1-k\, r^2}}\, dr.$$

In practical applications, there will be a minimum detectable flux, and we use the luminosity distance to determine the distance to objects producing grater than that flux, giving r_0. Calculation shows that almost all sources in such a count will be at distances close to r_0. In principle, source counts should give information that can be used to distinguish between cosmological models, but in practice it is difficult to separate the effects of the underlying cosmology from that of the evolution of the sources.